SET PROPERTIES DETERMINED BY CONDITIONS ON LINEAR SECTIONS

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Let $\mathcal{R}_n (n \geqq 2)$ be an n-dimensional Euclidean space, and let S be any set of points in $\mathcal{R}_n$. There exist a number of instances in which the following question has an interesting answer. Suppose a property A holds on each $(n-1)$-dimensional linear section $S_{n-1}{}^i$ of S. What additional property B assumed to hold on each section $S_{n-1}{}^i$ will insure that property A holds on S?

The following terminology is used. A continuum is a compact connected set which may include the degenerate case of a single point. Also compactness includes closure. A generalized continuum is a set which is connected and closed. An $(n-r)$-dimensional *linear section* of a set S with an $(n-r)$-dimensional Euclidean hyperplane L_{n-r} is defined to be the set $S \cdot L_{n-r}$. A subscript will *always* designate the dimensionality of the set.

1. Theorems on closed, open and bounded sets.

The following theorem illustrates the theory, and plays an important role in a succeeding theorem. It is a case in which condition B is sufficient but not necessary. We shall always assume $n \geqq 2$.

THEOREM 1. *Let S be any set in $\mathcal{R}_n$ $(n \geqq 2)$. If each $(n-1)$-dimensional linear section of S is connected and closed, then S is closed.*

PROOF. Suppose S is *not* closed. Then there exists a point $p \notin S$ which is a limit point of S. Let L_{n-1} be an $(n-1)$-dimensional hyperplane containing p, such that $S \cdot L_{n-1} \neq 0$. Since, by hypothesis, $S_{n-1} \equiv S \cdot L_{n-1}$ is closed, there exists an $(n-1)$-dimensional closed cube $C_{n-1} \subset L_{n-1}$, which contains p in its interior, and for which $C_{n-1} \cdot S_{n-1} = 0$. Let P_n be an n-dimensional hyperprism passing through C_{n-1}, and perpendicular to L_{n-1}. Since p is a limit point of S which is not in S, and since S_{n-1} is closed, there exists a sequence of points $p^i \in S \cdot P_n$, such that $p^i \notin L_{n-1}$, and such that $p^i \rightarrow p$ as $i \rightarrow \infty$. Let L_{n-2} be any $(n-2)$-dimensional hyperplane contained in L_{n-1} such that $S \cdot L_{n-2} \neq 0$, and such that $L_{n-2} \cdot C_{n-1} = 0$. Then there exists a sequence of hyperplanes $L_{n-1}{}^i$ determined by L_{n-2} and p^i. By hypothesis each set $S \cdot L_{n-1}{}^i$ is connected. Hence since $p^i \in S \cdot L_{n-1}{}^i \cdot P_n$, and since any point $q \in S \cdot L_{n-2} \cdot L_{n-1}{}^i$ is not in P_n, the connectedness of $S \cdot L_{n-1}{}^i$ im-

Presented to the Society, November 24, 1945; received by the editors November 16, 1945, and, in revised form, May 10, 1946.