SPACES CONGRUENT WITH BOUNDED SUBSETS OF THE LINE

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This note is devoted to a characterization of bounded subsets of the line from the viewpoint of metric geometry. Such spaces have been characterized by Menger and others,[1] usually from the consideration of conditions on imbeddability of finite subsets. The theorem given below differs from these in making use of a minimality property of the linear metric as the condition for congruence.

THEOREM. *Let S be a metric space with distance function $d(xy)$ and diameter t, possibly infinite. A necessary and sufficient condition that S be congruent with a bounded subset of the real line is that if $d'(xy)$ is another metric for S which is topologically equivalent*[2] *to $d(xy)$ and in which S has diameter t, then there are two points a and b of S such that $d'(ab) > d(ab)$.*

PROOF. The condition is necessary. For suppose that S is a bounded subset of the real line with diameter t, and that there exists a metric $d'(xy)$ for S such that (1) for some two points a and b, $d'(ab) < d(ab)$, $d(xy)$ denoting the Euclidean metric for S; (2) for each two points x and y, $d'(xy) \leqq d(xy)$; and (3) there exist sequences $\{x_n\}$, $\{y_n\}$, of points such that $d'(x_ny_n)$ approaches t as n increases. From (2), $\lim d(x_ny_n)$ is also t. There is no loss of generality in assuming that

$$\begin{aligned} d(x_ny_n) &= d(x_na) + d(ab) + d(by_n) \\ &= d(x_na) + d'(ab) + d(by_n) + [d(ab) - d'(ab)] \\ &\geqq d'(x_na) + d'(ab) + d'(by_n) + [d(ab) - d'(ab)] \\ &\geqq d'(x_ny_n) + [d(ab) - d'(ab)]. \end{aligned}$$

By taking the limit of both sides of this last inequality, we get an immediate contradiction of (1).

The condition is sufficient. There exist sequences $\{x_n'\}$, $\{y_n'\}$, of points of S such that $\lim d(x_n'y_n') = t$. Let $\{x_n\}$ be any subsequence of $\{x_n'\}$. For each two points x and y let

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[1] For definitions and bibliography, see L. M. Blumenthal, *Distance geometries*, University of Missouri Studies, 1938. This theorem had its beginning in discussions of some problems in measure with Dr. Dorothy Maharam.

[2] That is, in order that $d(x_nx)$ approach zero, it is necessary and sufficient that $d'(x_nx)$ approach zero.