SEMI-GROUPS OF OPERATORS AND THE WEIERSTRASS THEOREM

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In this note we give a short proof of the generalization[1] of M. H. Stone's representation of groups of unitary operators in Hilbert space and show how it yields the theorem of Weierstrass on uniform approximation by polynomials. This classical result in turn yields fairly easily the more recent algebraic-topological formulation of the Weierstrass theorem as given by M. H. Stone for real algebras and by I. Gelfand and G. Šilov[2] for complex algebras.

For $0 \leqq s < \infty$ let T_s be a linear operation in the real or complex Banach space X satisfying the conditions

$$(1) \qquad T_{s+t} = T_s T_t, \quad T_0 = I, \quad |T_s| \leqq 1, \quad \lim_{s \to t} T_s x = T_t x, \quad x \in X.$$

Let $A_h = h^{-1}[T_h - I]$ and $D(A)$ be the domain of definition of the operator $Ax \equiv \lim_{h \to 0} A_h x$.

THEOREM 1. *If T_s, $0 \leqq s < \infty$, is a semi-group of operators in X satisfying* (1) *then $D(A)$ is dense in X and uniformly for s in any finite interval we have*

$$(2) \qquad T_s x = \lim_{h \to 0} e^{sA_h} x, \qquad x \in X.$$

Note that $x_s \equiv s^{-1} \int_0^s T_u x du \to x$ and that

$$A_h x_s = (sh)^{-1} \left[\int_s^{s+h} T_u x du - \int_0^h T_u x du \right] \to A_s x \quad \text{as} \quad h \to 0.$$

Thus $D(A)$ is dense in X. Since $|e^{sA_h}| = |e^{(s/h)T_h} e^{-s/h}| \leqq e^{s/h} e^{-s/h} = 1$ it suffices to prove (2) for $x \in D(A)$. If $x \in D(A)$ we have

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[1] See I. Gelfand, C. R. (Doklady) Acad. Sci. URSS. vol. 25 (1939) pp. 713–718; M. Fukamiya, Proc. Imp. Acad. Tokyo vol. 16 (1940) pp. 262–265; E. Hille, Proc. Nat. Acad. Sci. U.S.A. vol. 28 (1942) pp. 175–178; E. Hille, ibid. pp. 421–424. The particular form of the theorem given here is that of E. Hille.

[2] M. H. Stone, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 375–481; I. Gelfand and G. Šilov, Rec. Math. (Mat. Sbornik) N.S. vol. 9 (1941) pp. 25–39. Our result differs from theirs in applying to spaces which are locally compact, and to subalgebras which are not assumed to contain a unit.