THE BOUNDEDNESS OF ORTHONORMAL POLYNOMIALS ON CERTAIN CURVES OF THE THIRD DEGREE

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1. **Introduction.** Properties of boundedness of systems of orthonormal polynomials, significant because of their relation to the convergence of the corresponding developments of "arbitrary" functions in series, constitute a subject for detailed investigation in themselves. In the case of the orthonormal systems associated with algebraic curves no method appears to be readily available for dealing with the problem as formulated in general terms, while on the other hand special methods of some degree of variety throw light on the facts relating to curves of particular types [6, 7, 4, 5].[1] An earlier paper by the present writer [4] is concerned with loci of the second degree; a more recent one [5] includes reference to the curves of the third degree represented by the equations $y=Ax^3+Bx^2+Cx+D$ and $y^2=x^3$. Although indefinite multiplication of particular instances would be unprofitable, some additional illustrations may serve to suggest a more adequate picture of the general situation. The following paragraphs present extensions of the reasoning to other curves with equations of the form $y^2=F(x)$, where $F(x)$ is a polynomial of the third degree.

2. **The curve** $y^2=x^2(x+1)$. This curve has a double point with two real branches intersecting at the origin. It has a parametric representation in which the coordinates are not merely rational functions but more specifically polynomials in the auxiliary variable,

$$x = t^2 - 1, \qquad y = t(t^2 - 1).$$

Since $x^3=y^2-x^2$, any monomial in terms of the coordinates containing x^3 as a factor can be replaced on the curve by an expression of lower degree. A fundamental sequence of monomials for application of the Schmidt process in constructing the orthogonal system on the curve is

$$(1) \qquad 1,\ x,\ y,\ x^2,\ xy,\ y^2,\ x^2y,\ xy^2,\ y^3,\ x^2y^2,\ xy^3,\ y^4,\ \cdots .$$

In terms of the parameter these are respectively

$$(2) \qquad 1,\ t^2 - 1,\ t(t^2 - 1),\ (t^2 - 1)^2,\ t(t^2 - 1)^2,\ t^2(t^2 - 1)^2,\ \cdots .$$

An arbitrary linear combination of a finite number of the above

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[1] Numbers in brackets refer to the references cited at the end of the paper.