NOTE ON NORMAL NUMBERS

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D. G. Champernowne[1] proved that the infinite decimal

$$0.123456789101112\cdots$$

was normal (in the sense of Borel) with respect to the base 10, a normal number being one whose digits exhibit a complete randomness. More precisely a number is normal provided each of the digits $0, 1, 2, \cdots, 9$ occurs with a limiting relative frequency of $1/10$ and each of the 10^k sequences of k digits occurs with the frequency 10^{-k}. Champernowne conjectured that if the sequence of all integers were replaced by the sequence of primes then the corresponding decimal

$$0.12357111317\cdots$$

would be normal with respect to the base 10. We propose to show not only the truth of his conjecture but to obtain a somewhat more general result, namely:

THEOREM. *If $a_1, a_2, \cdots$ is an increasing sequence of integers such that for every $\theta<1$ the number of a's up to N exceeds N^θ provided N is sufficiently large, then the infinite decimal*

$$0.a_1a_2a_3\cdots$$

is normal with respect to the base β in which these integers are expressed.

On the basis of this theorem the conjecture of Champernowne follows from the fact that the number of primes up to N exceeds $cN/\log N$ for any $c<1$ provided N is sufficiently large. The corresponding result holds for the sequence of integers which can be represented as the sum of two squares since every prime of the form $4k+1$ is also of the form x^2+y^2 and the number of these primes up to N exceeds $c'N/\log N$ for sufficiently large N when $c'<1/2$.

The above theorem is based on the following concept of Besicovitch.[2]

DEFINITION. *A number A (in the base β) is said to be (ϵ, k) normal if any combination of k digits appears consecutively among the digits of A with a relative frequency between $\beta^{-k}-\epsilon$ and $\beta^{-k}+\epsilon$.*

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[1] J. London Math. Soc. vol. 8 (1933) pp. 254–260.

[2] Math. Zeit. vol. 39 (1935) pp. 146–147.