RECIPROCALS OF J-MATRICES

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1. **Introduction.** We consider J-matrices

$$J = (j_{pq}), \qquad j_{pq} = 0 \quad \text{for} \quad |p - q| \geqq 2, \qquad j_{pp} = b_p,$$
$$j_{p+1,p} = j_{p,p+1} = -a_p \neq 0,$$

such that

$$(1.1)\quad I[J(x, \bar{x})] = \sum I(b_p) | x_p |^2 - \sum I(a_p)(x_p\bar{x}_{p+1} + \bar{x}_p x_{p+1}) \geqq 0$$

for all x_p for which the sums converge. These are the J-matrices associated with a positive definite J-fraction [**4**, **5**, **1**].[1] Let $X_p(z)$ and $Y_p(z)$ denote the solutions of the system of linear equations

$$(1.2)\quad -a_{p-1}x_{p-1} + (b_p + z)x_p - a_p x_{p+1} = 0, \; p = 1, 2, 3, \cdots; a_0 = 1,$$

under the initial conditions $x_0 = -1$, $x_1 = 0$ and $x_0 = 0$, $x_1 = 1$, respectively. We shall prove that when at least one of the series

$$(1.3)\qquad \sum_{p=1}^{\infty} | X_p(0) |^2, \qquad \sum_{p=1}^{\infty} | Y_p(0) |^2$$

diverges, then the matrix $J + zI$ has a unique bounded reciprocal for $I(z) > 0$, and that when both the series (1.3) converge then the matrix $J + zI$ has infinitely many different bounded reciprocals. This theorem was proved by Hellinger [2] for the case where the coefficients a_p and b_p are all real.

2. **Reciprocals of an arbitrary J-matrix.** The general right reciprocal of $J + zI$ is (ρ_{pq}) where $\rho_{1,q}$, $q = 1, 2, 3, \cdots$, are arbitrary functions of z, and [**3**, p. 116]

$$(2.1)\qquad \rho_{pq}(z) = \begin{cases} \rho_{1,q}(z) Y_p(z), & p = 1, 2, 3, \cdots, q; \\ \rho_{1,q}(z) Y_p(z) + X_q(z) Y_p(z) - X_p(z) Y_q(z), & \\ & p = q+1, q+2, q+3, \cdots. \end{cases}$$

We shall say that the *determinate case* or the *indeterminate case* holds for the J-matrix according as at least one of the series (1.3) diverges or both of these series converge, respectively. In the indeterminate case, both of the series

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.