ALMOST PERIODICITY, EQUI-CONTINUITY AND TOTAL BOUNDEDNESS

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Let X be a uniform space; that is to say, let X be a space provided with a system of indexed neighborhoods $U_\alpha(x)$ ($x \in X$, α = index), subject to the conditions (A. Weil): (1) If $x \in X$ and if α is an index, then $x \in U_\alpha(x)$; (2) If α and β are indices, then there exists an index γ such that $x \in X$ implies $U_\gamma(x) \subset U_\alpha(x) \cap U_\beta(x)$; (3) If α is an index, then there exists an index β such that $x, y, z \in X$ with $x, y \in U_\beta(z)$ implies $x \in U_\alpha(y)$. Let T be a topological group with identity σ and let f be a transformation of $X \times T$ into X. We agree to write $f^t(x)$ or $f_x(t)$ in place of $f(x, t)$ ($x \in X$, $t \in T$), whenever we wish. Furthermore, let f define a transformation group; that is to say, suppose $f^\sigma(x) = x$ and $f^s f^t(x) = f^{ts}(x)$ ($x \in X$; $t, s \in T$). We impose continuity conditions on f as the needs arise.

A subset E of T is said to be *relatively dense* provided there exists a compact set A in T such that each left translate of A intersects E. A point x of X is called *almost periodic* provided that if U is a neighborhood of x, then there exists a relatively dense set E in T for which $f(x, E) \subset U$. We observe that the notion of almost periodic point depends on the topology in T, the strongest type of almost periodicity occurring when T is provided with the discrete topology. It is easily proved that a set E in T is relatively dense if and only if there exists a compact set B in T such that $T = EB$. The set $f(x, T)$ is called the *orbit* of the point x.

THEOREM 1. *If the family $[f^t \mid t \in T]$ is equi-continuous at x, if f_x is continuous on T, and if x is almost periodic, then the orbit of x is totally bounded. Conversely, if the family $[f^t \mid t \in T]$ is equi-uniformly continuous and if the orbit of x is totally bounded, then x is almost periodic.*

PROOF. Suppose the hypotheses of the first statement hold. Let α be an index. There exists an index β such that the β-neighborhood of each compact set in X is contained in the union of finitely many α-neighborhoods. By hypothesis we can find an index γ such that $f^t(U_\gamma(x)) \subset U_\beta(f^t(x))$ ($t \in T$). There are sets E and A in T such that $T = EA$, A is compact, and $f(x, E) \subset U_\gamma(x)$. Hence, $f(x, T) \subset f(U_\gamma(x), A) \subset U_\beta(f(x, A))$. Since $f(x, A)$ is compact, $U_\beta(f(x, A))$ is contained in

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