# THE ASANO POSTULATES FOR THE INTEGRAL DOMAINS OF A LINEAR ALGEBRA

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1. **Introduction.** The multiplicative ideal theory for a noncommutative ring $A$ as developed by Asano[1] postulates the existence in $A$ of a maximal bounded order $R$ which satisfies the maximal chain condition for two-sided $R$-ideals contained in $R$ and the minimal chain condition for one-sided $R$-ideals in $R$ containing any fixed two-sided $R$-ideal. Let $A$ be a separable algebra over the field $P$, and let $P$ be the quotient field of the domain of integrity $g$. It has been shown [2, pp. 123–126] that if $g$ has a Noether ideal theory, then a maximal domain of $g$-integers exists in $A$ and satisfies the conditions of the Asano theory. It is the purpose of this paper to prove that the condition of separability can be removed from $A$ and that it need only be postulated that $A$ shall have an identity.

2. **Subgroups of direct sums.** Let $G$ be a commutative group with operator domain $\Omega$. Let $G$ be the direct sum of the $\Omega$-subgroups $G_1, G_2, \cdots, G_n$. We shall write $G=G_1+G_2+\cdots+G_n$. The direct summand $G_i$ gives rise to a projection $\alpha_i$ which is an endomorphism of $G$ on $G_i$: if $g=g_1+g_2+\cdots+g_n$, $g_j\in G_j$, then $\alpha_i g=g_i$. The sum $\alpha_1+\alpha_2+\cdots+\alpha_n$ is the identity operator $I$. Furthermore the sum of any subset of the projections $\alpha_1, \alpha_2, \cdots, \alpha_n$ is a projection. We shall label in particular the operators $\delta_i=\sum_{j=1}^{i}\alpha_j$. Then $\delta_1=\alpha_1$, and $\delta_n=I$. In general $\delta_{i+1}=\delta_i+\alpha_{i+1}$. If $\omega\in\Omega$, then $\omega\alpha_i=\alpha_i\omega$, and as a result $\omega\delta_i=\delta_i\omega$; that is, $\alpha_i$ and $\delta_i$ are $\Omega$-operators. It follows that $\alpha_iH$ and $\delta_iH$ are $\Omega$-subgroups whenever $H$ is an $\Omega$-subgroup.

LEMMA 1. *Let the commutative group $G=G_1+G_2+\cdots+G_n$ contain the $\Omega$-subgroups $H$ and $K$. If $H\supseteq K$, then $\alpha_iH\supseteq\alpha_iK$, $\delta_iH\supseteq\delta_iK$, and $\delta_iH\cap G_i\supseteq\delta_iK\cap G_i$.*

Since $H\supseteq K$, the image $\alpha_iK$ of $K$ under the homomorphism of $H$ on $\alpha_iH$ must be contained in $\alpha_iH$. By the same argument $\delta_iH\supseteq\delta_iK$, and therefore $\delta_iH\cap G_i\supseteq\delta_iK\cap G_i$.

LEMMA 2. *Let the commutative group $G=G_1+G_2+\cdots+G_n$ contain the $\Omega$-subgroups $H$ and $K$. If $H\supseteq K$ and if $\alpha_iH=\alpha_iK$, $\delta_iH\cap G_i=\delta_iK\cap G_i$, then $H=K$.*

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[1] Cf. Asano [1], Jacobson [2]. We use here the formulation of these postulates given by Jacobson. Numbers in brackets refer to the references at the end of the paper.