ON DEGREE OF APPROXIMATION ON A JORDAN CURVE TO A FUNCTION ANALYTIC INTERIOR TO THE CURVE BY FUNCTIONS NOT NECESSARILY ANALYTIC INTERIOR TO THE CURVE

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It is our object here to consider the subject mentioned in the title by proving the following theorem.

THEOREM. *Let C be a rectifiable Jordan curve in the plane of the complex variable z, and let the function $f(z)$ be analytic interior to C, continuous in the corresponding closed region. Let each of the functions $f_n(z)$, $n=1, 2, \cdots$, be analytic exterior to C and continuous in the corresponding closed region, except perhaps for poles of total order not greater than n exterior to C. We write*

$$f_n(z) \equiv g_n(z) + h_n(z), \tag{1}$$

for z on and exterior to C, where $g_n(z)$ is a rational function of z with all its poles exterior to C and $h_n(z)$ is analytic in the extended plane exterior to C, continuous in the corresponding closed region, with $h_n(\infty)=0$. If the totality of poles of all the $f_n(z)$ have no limit point on C, and if we have

$$\limsup_{n\to\infty} [\max |f(z) - f_n(z)|, z \text{ on } C]^{1/n} \leqq 1/R < 1, \tag{2}$$

then we have also

$$\limsup_{n\to\infty} [\max |f(z) - g_n(z)|, z \text{ on } C]^{1/n} \leqq 1/R, \tag{3}$$

$$\limsup_{n\to\infty} [\max |h_n(z)|, z \text{ on } C]^{1/n} \leqq 1/R. \tag{4}$$

The significance of this theorem is in part as follows. In numerous situations a function $f(z)$ is approximated on a Jordan curve C by a function of type $f_n(z)$; for instance $f(z)$ may be approximated on the unit circle C: $|z|=1$ by a trigonometric polynomial in the arc length, of order n, which is of type $f_n(z)$. If this approximating function $f_n(z)$ is split into the two components $g_n(z)$ and $h_n(z)$, what does $h_n(z)$ contribute to the degree of approximation? This question is answered by the theorem, asserting that asymptotically $h_n(z)$ contributes nothing to the degree of approximation, insofar as approximation is measured by the first member of (2); it is no more favorable to approximate

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