A NOTE ON THE RIEMANN ZETA-FUNCTION

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Let $\rho_\nu = \beta_\nu + i\gamma_\nu$ be the zeros of the Riemann zeta-function $\zeta(1/2+z)$ whose real part $\beta_\nu \geqq 0$. Then we have the following formula which is an improvement on Paley-Wiener's [1, p. 78][1]

$$\int_1^T \frac{\log |\zeta(1/2+it)|}{t^2} dt = 2\pi \sum_{\nu=1}^{\infty} \frac{\beta_\nu}{|\rho_\nu|^2} + \int_0^{\pi/2} R\{e^{-i\theta} \log \zeta(1/2+e^{i\theta})\} d\theta + O\left(\frac{\log T}{T}\right).$$

In order to prove this formula let ρ_ν ($\nu = 1, 2, \cdots, n$) be the n zeros of $\zeta(1/2+z)$ for which $0<\gamma_\nu<T$ and $0 \leqq \beta_\nu < 1/2$. We require the following lemma:

LEMMA. *Let K be the unit semicircle with center $z=0$ lying in the right half-plane $R(z)>0$ and let C be the broken line consisting of three segments L_1 $(0 \leqq x \leqq T,\ y=T)$, L_2 $(0 \leqq x \leqq T,\ y=-T)$ and L_3 $(x=T,\ -T \leqq y \leqq T)$. Then*

$$\frac{1}{\pi}\int_1^T \frac{\log |\zeta(1/2+it)|}{t^2} dt = 2\sum_{\nu=1}^{n} \frac{\beta_\nu}{|\rho_\nu|^2} + \frac{1}{2\pi i}\int_K \frac{\log \zeta(1/2+z)}{z^2} dz - \frac{1}{2\pi i}\int_C \frac{\log \zeta(1/2+z)}{z^2} dz. \tag{1}$$

This is a form of Carleman's theorem which can be proved by a method of proof analogous to that of Littlewood's theorem (Titchmarsh [3, pp. 130–134]).

Let Γ be a contour describing C, K and the corresponding part of the imaginary axis, and let ρ_ν be a point interior to Γ, and $\log(z-\rho_\nu)$ be taken as its principal value. We write C_1 as a contour describing Γ in positive direction to the point $i\gamma_\nu$, then along the segment $y=\gamma_\nu$, $0<x<\beta_\nu - r$, and describing a small circle with center $z=\rho_\nu$, radius r, then going back along the negative side of this segment to $i\gamma_\nu$, and then along Γ to the starting point.

By Cauchy's theorem we get

$$\int_{C_1} \frac{\log (z-\rho_\nu)}{z^2} dz = 0.$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.