ON CERTAIN LIMIT THEOREMS OF THE THEORY OF PROBABILITY

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1. Introduction. In this paper we prove the following four limit theorems:

Let $X_1, X_2, X_3, \cdots$ be independent identically distributed random[1] *variables each having mean* 0 *and standard deviation* 1. *Let*

$$s_k = X_1 + X_2 + \cdots + X_k,$$

then:

I.

$$\lim_{n\to\infty} \text{prob. } \{\max(s_1, s_2, \cdots, s_n) < \alpha n^{1/2}\} = \sigma_1(\alpha)$$

where

$$\sigma_1(\alpha) = 0 \qquad (\alpha \leqq 0)$$

and

$$\sigma_1(\alpha) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\alpha \exp\left(-\frac{u^2}{2}\right) du \qquad (\alpha \geqq 0).$$

II.

$$\lim_{n\to\infty} \text{prob. } \{\max(|s_1|, |s_2|, \cdots, |s_n|) < \alpha n^{1/2}\} = \sigma_2(\alpha)$$

where

$$\sigma_2(\alpha) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \exp\{-(2m+1)^2\pi^2/8\alpha^2\} \quad (\alpha \geqq 0).$$

III.

$$\lim_{n\to\infty} \text{prob. } \left\{\frac{s_1^2 + s_2^2 + \cdots + s_n^2}{n^2} < \alpha\right\} = \sigma_3(\alpha),$$

where

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[1] This condition can be replaced by a weaker one. In fact, it is enough to assume that the X's are such that the central limit theorem is applicable.