PRIME IDEALS AND INTEGRAL DEPENDENCE

I. S. COHEN AND A. SEIDENBERG

Let $\mathfrak{R}$ and $\mathfrak{S}$ be commutative rings such that $\mathfrak{S}$ contains, and has the same identity element as, $\mathfrak{R}$. If $\mathfrak{p}$ and $\mathfrak{P}$ are prime ideals in $\mathfrak{R}$ and $\mathfrak{S}$ respectively such that $\mathfrak{P} \cap \mathfrak{R} = \mathfrak{p}$ then we shall say that $\mathfrak{P}$ lies over, or contracts to, $\mathfrak{p}$. If over every prime ideal in $\mathfrak{R}$ there lies a prime ideal in $\mathfrak{S}$, we shall say that the "lying-over" theorem holds for the pair of rings $\mathfrak{R}$ and $\mathfrak{S}$.

Suppose now that $\mathfrak{q}$ and $\mathfrak{p}$ are prime ideals in $\mathfrak{R}$ such that $\mathfrak{q} \subset \mathfrak{p}$. If for every prime ideal $\mathfrak{Q}$ in $\mathfrak{S}$ lying over $\mathfrak{q}$ there exists a prime ideal $\mathfrak{P}$ in $\mathfrak{S}$ lying over $\mathfrak{p}$ and containing $\mathfrak{Q}$, then the "going-up" theorem will be said to hold for $\mathfrak{R}$ and $\mathfrak{S}$. Similarly, if for every prime ideal $\mathfrak{P}$ in $\mathfrak{S}$ lying over $\mathfrak{p}$ there exists a prime ideal $\mathfrak{Q}$ in $\mathfrak{S}$ lying over $\mathfrak{q}$ and contained in $\mathfrak{P}$, then the "going-down" theorem will be said to hold.

Below we are concerned with the case where $\mathfrak{S}$ is integrally dependent on $\mathfrak{R}$. In this case we shall prove the "lying-over" and "going-up" theorems (§1). With certain additional conditions on $\mathfrak{R}$ and $\mathfrak{S}$, also the "going-down" theorem is proved (§2). Counterexamples are given to show that none of these conditions can be omitted (§3).

All of the results of this paper (except Theorem 7) have been proved by Krull[1] when the rings are free from zero-divisors. The present proofs are essentially simpler than Krull's and at the same time do not require that the rings be integral domains.

1. The "lying-over" and "going-up" theorems. Let $\mathfrak{R}$ and $\mathfrak{S}$ be commutative rings with $\mathfrak{R}$ contained in $\mathfrak{S}$ and with a common identity element, and let $\mathfrak{S}$ be integral over $\mathfrak{R}$. We examine first the question of whether the "lying-over" theorem holds for the rings $\mathfrak{R}$ and $\mathfrak{S}$. We remark that if a maximal ideal in $\mathfrak{S}$ necessarily contracts to a maximal ideal in $\mathfrak{R}$, and if $\mathfrak{R}$ has a single maximal ideal $\mathfrak{p}$, then for the prime ideal $\mathfrak{p}$ it is certainly true that there exists a prime ideal in $\mathfrak{S}$ lying over $\mathfrak{p}$; in fact, every maximal ideal of $\mathfrak{S}$ will lie over $\mathfrak{p}$. This remark is the motivation behind the following theorem.

THEOREM 1. *Let $\mathfrak{S}$ be integral over $\mathfrak{R}$, and let the prime ideal $\mathfrak{P}$ in $\mathfrak{S}$ lie over the prime ideal $\mathfrak{p}$ in $\mathfrak{R}$, that is, $\mathfrak{P} \cap \mathfrak{R} = \mathfrak{p}$. Then $\mathfrak{p}$ is maximal if and only if $\mathfrak{P}$ is maximal.*

Presented to the Society, September 15, 1945; received by the editors October 31, 1945.

[1] *Zum Dimensionsbegriff der Idealtheorie*, Math. Zeit. vol. 42 (1937) pp. 745–766. Especially relevant are Theorems 1–6 and the considerations on pp. 756–757.