NOTE ON A NOTE OF H. F. TUAN

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The following theorem is proved.

THEOREM. *If Z is a nilpotent matrix with elements in a field K, then the replicas of Z are those and only those matrices which are of the form $f(Z)$, where $f(x)$ is an additive*[1] *polynomial in $K[x]$.*

The concept of a replica was introduced by Chevalley,[2] who proved this theorem when K is of characteristic zero. The theorem was proved in general by H. F. Tuan[3] by elementary methods. The object of this note is to give a simplification of Tuan's proof; in particular, computations involving the specific form of Z are avoided.

If $h(x)$ is additive, then according as K is of characteristic 0 or p, $h(x)$ will have one of the two forms

$$tx, \qquad \sum_{j=1}^{m} t_j x^{p^j} \qquad (t, t_j \in K). \tag{1}$$

For if $h(x)$ had any other terms, then $h(x)+h(y)=h(x+y)$ would contain product terms $x^\alpha y^\beta$, $\alpha>0$, $\beta>0$. Conversely, polynomials of the form (1) are clearly additive. If $h(x)=\sum_{k=0}^{s} c_k x^k (c_k \in K)$, then we define

$$h^{[i]}(x) = \sum_{k=i}^{s} C_{k,i} c_k x^{k-i},$$

where the $C_{k,i}$ are binomial coefficients. Evidently

$$h^{(i)}(x) = i! h^{[i]}(x), \qquad h(x+y) = \sum_{i=0}^{s} h^{[i]}(x) y^i.$$

It follows from this that $h(x)$ is additive if and only if $c_0=0$ and $h^{[i]}(x)=c_i$ for $i>0$.

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[1] A polynomial $f(x)$ is additive if $f(x+y)=f(x)+f(y)$. The statement of the theorem in terms of the additivity of $f(x)$ rather than in terms of the explicit form (1), as well as the use of the derived polynomials $f^{[i]}(x)$ to replace explicit computation with binomial coefficients, was suggested by Professor Jacobson.

[2] Claude Chevalley, *A new kind of relationship between matrices*, Amer. J. Math. vol. 65 (1943) pp. 521–531. We make use of the definitions and notations of this paper.

[3] Hsio-Fu Tuan, *A note on the replicas of nilpotent matrices*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 305–312, in particular Theorems (A) and (D).