REGULARITY PROPERTIES OF A CERTAIN CLASS OF SURFACES

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Introduction. Convex curves in the plane and convex surfaces in space have many at first sight unexpected regularity properties.[1] The stringency of the convexity condition makes it desirable to find a geometric property of convex curves or surfaces which has similar analytical implications but applies to a wider class of geometrical objects. The first and most studied generalization of convexity is finiteness of the order: a curve or surface is said to be of order n if its intersection with any straight line consists of at most n connected sets[2] (segments, points, or rays). However, this condition proves to be much too weak.[3] On the other hand, the dual condition, finiteness of class, will turn out to be more restrictive than necessary.

The present paper will show that for curves *finiteness of the total curvature* is an entirely satisfactory condition. It includes in particular all curves of finite class. More generally, hypersurfaces in E^n with finite total curvature seem to share the most important differentiability properties with convex hypersurfaces.

The present results on curves with finite total curvature and the modern theory of functions of a real variable lead to the following surprising fact concerning surfaces in E^3: *If the paratingens*[4] *of Φ at a given point p leaves out at least one line L, and if, locally, the plane sections of Φ parallel to L have uniformly bounded total curvature, then the following facts hold simultaneously at almost all points q of Φ: the surface Φ has a tangent plane Π at q. All plane sections ($\neq\Pi$) of Φ through p have a curvature*[5] *at p, and these curvatures satisfy the theorems of Meusnier and Euler.*

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[1] For curves compare Jessen [1], for surfaces Busemann-Feller [1], and for hypersurfaces Alexandrov [1]. Numbers in brackets refer to the references cited at the end of the paper.

[2] This formulation is due to Hjelmslev [2] and has the obvious advantage not to exclude polygons, polyhedrons, ruled surfaces, $\cdots$.

[3] Compare Marchaud [1], Haupt [1], and §2 of the present paper.

[4] This concept is due to Bouligand, see Bouligand [1]. The paratingens of Φ represented parametrically by $p(u, v)$ at (u_0, v_0) consists of all non-oriented lines G which are limits of sequences of non-oriented lines $G(p(u_\nu', v_\nu'), p(u_\nu, v_\nu))$ with $(u_\nu', v_\nu') \neq (u_\nu, v_\nu)$ and $u_\nu', u_\nu \to u_0$, $v_\nu', v_\nu \to v_0$. For the definition of total curvature see the following §1.

[5] More precisely the ordinary curvature as defined at the end of §1.