ON A SPECIAL CLASS OF ABELIAN FUNCTIONS

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In the applications of the theory of Riemann matrices to algebraic geometry and arithmetic it is necessary to emphasize the rings of complex multiplications in preference to the rational algebra associated to a given Riemann matrix. In recent years A. A. Albert completed and amplified the algebraic theory of Riemann matrices that was inaugurated by G. Scorza, S. Lefschetz, and others.[1] We shall construct in this note the fields of abelian functions of genus n whose multiplication rings contain a given order of a totally real field of absolute degree n. For the proof we shall compare two types of Riemann matrices defined by S. Lefschetz and O. Blumenthal.[2] Applying equivalences to these Riemann matrices we find suitable matrices which show explicitly how a permissible ring of complex multiplications can be realized.

Let $k=R(\alpha)$ be a totally real field of degree n. Suppose that $\alpha=\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(n)}$ are the n conjugate roots of the defining irreducible equation $f(\alpha)=0$. Lefschetz[3] has determined the form of the most general Riemann matrix Ω_1 of genus n whose multiplication algebra $\mathfrak{A}(\Omega_1)$ contains an isomorphic image of the field k. The matrix Ω_1 is equal to

$$\left\|\begin{matrix} \tau_{11}, & \tau_{11}\alpha^{(1)}, & \cdots, & \tau_{11}\alpha^{(1)n-1}; & \tau_{12}, & \tau_{12}\alpha^{(1)}, & \cdots, & \tau_{12}\alpha^{(1)n-1} \\ \tau_{21}, & \tau_{21}\alpha^{(2)}, & \cdots, & \tau_{22}\alpha^{(2)n-1}; & \tau_{22}, & \tau_{22}\alpha^{(2)}, & \cdots, & \tau_{22}\alpha^{(2)n-1} \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \tau_{n1}, & \tau_{n1}\alpha^{(n)}, & \cdots, & \tau_{n1}\alpha^{(n)n-1}; & \tau_{n2}, & \tau_{n2}\alpha^{(n)}, & \cdots, & \tau_{n2}\alpha^{(n)n-1} \end{matrix}\right\|,$$

where $\tau_{11}, \cdots, \tau_{n1}, \tau_{12}, \cdots, \tau_{n2}$ are complex parameters subject to the restriction that Ω_1 satisfy the defining equations of a Riemann matrix.

Now suppose that $\alpha_1=\alpha_1^{(1)}, \cdots, \alpha_n=\alpha_n^{(1)}$ is a basis of k/R. Let $A_i^{(j)}$ be the minors of the determinant $\delta=\delta^{(1)}=|\alpha_i^{(j)}|$. Then there ex-

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[1] For references see Solomon Lefschetz, *Selected topics on algebraic geometry*, National Research Council Bulletin vol. 63 (1928) pp. 310–395, and A. A. Albert, *Structure of Algebras*, Amer. Math. Soc. Colloquium Publications, vol. 24, 1939.

[2] Solomon Lefschetz, *On certain numerical invariants of algebraic varieties with applications to abelian varieties*, Trans. Amer. Math. Soc. vol. 22 (1921) pp. 327–482 (quoted as L); Otto Blumenthal, *Über Thetafunktionen und Modulfunktionen mehrerer Veränderlicher*, Jber. Deutschen Math. Verein. vol. 13 (1904) pp. 120–132 (quoted as B).

[3] L, loc. cit. p. 397.