ORBIT-CLOSURE DECOMPOSITIONS AND ALMOST PERIODIC PROPERTIES

W. H. GOTTSCHALK

Let X be a metric space with metric ρ, let $f(X)\subset X$ be a continuous mapping, and let $h(X)=X$ be a homeomorphism. For $x\in X$, the set $\sum_{n=0}^{+\infty} f^n(x)$ is called the *semi-orbit* of x under f and the set $_n\sum_{-\infty}^{+\infty} h^n(x)$ is called the *orbit* of x under h. For $x\in X$, the closure of the semi-orbit of x under f is called the *semi-orbit-closure* of x under f and the closure of the orbit of x under h is called the *orbit-closure* of x under h.

A nonvacuous subset Y of X is said to be *semi-minimal* (*minimal*) under $f(h)$ provided that the semi-orbit-closure (orbit-closure) of each point of Y is Y. Clearly, any two semi-minimal (minimal) sets are either coincident or disjoint. It is easily proved that a subset Y of X is semi-minimal (minimal) under $f(h)$ if and only if Y is nonvacuous, closed, $f(Y)\subset Y(h(Y)=Y)$, and furthermore Y contains no proper subset with these properties. We follow Birkhoff [**2**, p. 198][1] in the terminology of "minimal set."

A *decomposition* of X is defined to be a collection of nonvacuous closed pairwise disjoint subsets of X which fill up X. We say that the mapping f *gives a semi-orbit-closure* (*a semi-minimal set*) *decomposition* provided that the collection of semi-orbit-closures (semi-minimal sets) is a decomposition of X. Also, it is said that the homeomorphism h *gives an orbit-closure* (*a minimal-set*) *decomposition* provided that the collection of orbit-closures (minimal sets) is a decomposition of X.

A point x of X is said to be *almost periodic* under f provided that to each $\epsilon>0$ there corresponds a positive integer N with the property that in every set of N consecutive positive integers appears an integer n such that $\rho(x,f^n(x))<\epsilon$. The mapping f is said to be *pointwise almost periodic* provided that each point of X is almost periodic under f. It is to be noted that various writers use the above terms in different senses and employ other terminologies for these notions.

LEMMA 1. *The mapping f (homeomorphism h) gives a semi-orbit-closure (an orbit-closure) decomposition if and only if $f(h)$ gives a semi-minimal-set (a minimal-set) decomposition; and in either event, the two decompositions coincide.*

Presented to the Society, August 14, 1944; received by the editors May 15, 1944 and November 15, 1944.

[1] Numbers in brackets refer to the bibliography at the end of the paper.