CONGRUENCES INVOLVING THE PARTITION FUNCTION $p(n)$

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1. Introduction. The purpose of this note is to give certain congruence properties of $p(n)$, the partition function, for moduli 13 and 17, analogous to those obtained by Ramanujan for moduli 5, 7 and 11. The method and notation employed are essentially those of Ramanujan in [1].[1] Let

$$P = P(x) = 1 - 24\sum_{n=1}^{\infty} \frac{nx^n}{1 - x^n},$$

$$Q = Q(x) = 1 + 240\sum_{n=1}^{\infty} \frac{n^3x^n}{1 - x^n},$$

$$R = R(x) = 1 - 504\sum_{n=1}^{\infty} \frac{n^5x^n}{1 - x^n},$$

$$f(x) = \prod_{i=1}^{\infty} (1 - x^i).$$

Then it is known that

$$f(z) = 1 - x - x^2 + x^5 + x^7 - \cdots$$

$$= \sum_{n=-\infty}^{+\infty} (-1)^n x^{n(3n-1)/2}, \tag{1}$$

$$Q^3 - R^2 = 1728x[f(x)]^{24}. \tag{2}$$

Furthermore, let $\Phi_{r,s}(x) = \sum_{n=1}^{\infty} n^r\sigma_{s-r}(n)x^n$, where $\sigma_k(n)$ denotes the sum of the kth powers of the divisors of n. In particular

$$\Phi_{0,s}(x) = \sum_{n=1}^{\infty} \frac{n^sx^n}{1 - x^n} = \sum_{n=1}^{\infty} \sigma_s(n)x^n,$$

so that

$$P = 1 - 24\Phi_{0,1}(x),$$

$$Q = 1 + 240\Phi_{0,3}(x),$$

$$R = 1 - 504\Phi_{0,5}(x).$$

Then in terms of the functions $\sum_{r,s}(n)$, defined by

$$\sum\nolimits_{r,s}(n) = \sum_{m=0}^{n} \sigma_r(m)\sigma_s(n - m),$$

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[1] Numbers in brackets refer to the Bibliography at the end of the paper.