A NOTE ON A THEOREM OF HARDY ON FOURIER CONSTANTS

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In Note LXVI of his *Notes on some points in the integral calculus* [1],[1] Hardy proves the following theorem.

THEOREM 1 (HARDY). *If* $a_1, a_2, \cdots, a_n, \cdots$ *are the Fourier constants of a function of* L^p, $p \geqq 1$, *then* $A_1, A_2, \cdots, A_n, \cdots$ *are also the Fourier constants of a function of* L^p, *where* $A_n = n^{-1}\sum_1^n a_k$.

Hardy restricts the class of functions considered to even functions, with mean value over a period zero, and the same restriction shall be observed here.

We wish to prove a "dual" to Theorem 1, namely:

THEOREM 2. *If* $b_1, b_2, \cdots, b_n, \cdots$ *are the Fourier constants of a function,* $g(x)$, *of* L^p, $p>1$, *then* $B_1, B_2, \cdots, B_n, \cdots$ *are also the Fourier constants of a function,* $G(x)$, *of* L^p, *where* $B_n = \sum_n^\infty b_k k^{-1}$, *and if* $f(x) \sim \sum a_n \cos nx$ *is any function of* $L^{p'}$, $F(x) \sim \sum_1^\infty A_n \cos nx$, *then,* $1/p + 1/p' = 1$,

$$\int_0^\pi f(x)G(x)dx + \int_0^\pi F(x)g(x)dx = 0.$$

There is a difference between the theorems. The case $p=1$ is omitted in Theorem 2, and necessarily, since $g(x) \sim \sum_{n=2}^\infty \cos nx/\log n$ belongs to L, but the corresponding B_n do not even exist.

Hardy's method depends upon an explicit representation of the function $F(x)$ in terms of $f(x)$. In the case of the theorem to be proven, this representation does not seem to facilitate matters. The method used will depend upon some general theorems on Fourier series, together with the original theorem of Hardy. For the case $p=2$, the proof is immediate, since it can be shown that the convergence of b_n^2 implies the convergence of B_n^2 (this result is also due to Hardy, and was the origin of this type of theorem). In particular, this series converges for a function belonging to L^p, for $p \geqq 2$, and thus there exists a function of L^2, having the B_n as Fourier coefficients.

The method of proof depends upon the observation that, purely formally, partial summation yields

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[1] Numbers in brackets refer to the references cited at the end of the paper.