A NOTE ON AN INEQUALITY OF E. SCHMIDT

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In a note inserted in the Sitzungsberichte der Preussicher Akademie, 1932, E. Schmidt stated without proof a relation

$$\left[\int_{-1}^{1} f'(x)^2 dx\right]^{1/2} \leqq k_N N^2 \left[\int_{-1}^{1} f(x)^2 dx\right]^{1/2} \tag{1}$$

where $f(x)$ is a polynomial of degree N, and it was subsequently shown by Hille, Szego and Tamarkin that k_N is a bounded function of N such that $\lim_{N\to\infty} k_N = \pi^{-1}$.

In a joint paper, generalizing the classical Markoff theorem

$$\max_{-1\leqq x\leqq 1} |f'(x)| \leqq N^2 \max_{-1\leqq x\leqq 1} |f(x)|$$

to general mean values of the form $\int_{-1}^{1} |f(x)|^p dx$, $p\geqq 1$, Hille, Szego and Tamarkin [1][1] gave three proofs of the inequality (1), two products of the general case of exponent p, and another for the case $p=2$, stated by the authors to be similar to the original unpublished proof of Schmidt.

It seems worthwhile to sketch a short elementary proof of the important case $p=2$, depending only upon an elementary inequality:[2]

$$n\sum_{1}^{n} a_k^2 - \left(\sum_{1}^{n} a_k\right)^2 \geqq 0 \tag{2}$$

and a simple property of the Legendre polynomials.

The required result for Legendre polynomials is [2] $P'_{n+1}(z) - P_{n-1}(z) = (2n+1)P_n(z)$, where $P_n(z)$ is the nth Legendre polynomial. From this we obtain

$$\begin{aligned} P'_{2n} &= \sum_{0}^{n-1} (4k+3)P_{2k+1}, \\ P'_{2n+1} &= \sum_{0}^{n} (4k+1)P_{2k}. \end{aligned} \tag{3}$$

Let the polynomial of the Nth degree be expressed in Legendre polynomials

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] The use of this inequality, instead of a more complicated one due to Hardy, was suggested by Professor Szász. It simultaneously simplifies the proof and yields a better constant than that originally obtained by the author.