MATRIX PRODUCTS OF MATRIX POWERS

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1. Introduction. Let m n-by-n matrices, A_k, of complex constants, a_{ijk} $(i, j=1, 2, \cdots, n; k=1, 2, \cdots, m)$, be given. We shall denote by $\mathcal{L}$ the set of all matrices,

$$A(t) = \sum_{i=1}^{m} \rho_i(t)A_i,$$

where $\rho_i(t)$ $(i=1, 2, \cdots, m)$ are arbitrary, non-negative, summable functions of the real variable t on the interval T, $a \leqq t \leqq b$. We shall call $\mathfrak{J}$, $\mathfrak{S}$, or $\mathfrak{X}$ the subsets of $\mathcal{L}$ obtained by restricting the functions $\rho_i(t)$ to polynomial functions, step functions, or step functions which are all zero except one. Since, in each case, the elements of $A(t)$ are summable functions of t on T, it follows that, on T, there exists a unique, absolutely continuous matrix solution,[1] $Y(t)$, of the linear, matrix differential equation and initial condition:

$$dY(t)/dt = Y(t)A(t), \qquad Y(a) = E, \tag{1.1}$$

where E is the n-by-n unit matrix. We shall denote by λ, ι, σ or ξ the set of matrices, $Y(t)$, which are particular values of solutions of (1.1), where $A(t)$ is an arbitrary matrix of $\mathcal{L}$, $\mathfrak{J}$, $\mathfrak{S}$, or $\mathfrak{X}$, respectively, and t is on T.

If A is a matrix with elements a_{ij}, let the absolute value of A and the exponential and natural logarithm of A be defined[2] by the equations:

$$|A| = \left[\sum_{i,j=1}^{n} |a_{ij}|^2\right]^{1/2},$$

$$\exp A = \sum_{i=0}^{\infty} A^i/i!$$

$$\log A = \sum_{i=1}^{\infty} (-1)^{i-1}(A-E)^i/i, \quad \text{if} \quad |A-E| < 1.$$

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[1] See W. M. Whyburn, *On the fundamental existence theorems for differential systems*. Ann. of Math. (2) vol. 30 (1928–29) p. 31. We observe that equations (1.1) are equivalent to a system of $2n$ real, linear, first order differential equations satisfying all the hypotheses of this theorem.

[2] See J. v. Neumann, *Über die analytischen Eigenschaften von Gruppen linearer Transformationen und ihrer Darstellungen*. Math. Zeit. vol. 30 (1929) pp. 6, 7.