TWIN CONVERGENCE REGIONS FOR CONTINUED FRACTIONS

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1. Introduction. Some years ago it was proved [2][1] that the continued fraction

$$1+\frac{a_1}{1}+\frac{a_2}{1}+\cdots \tag{1.1}$$

converges if the complex numbers a_n satisfy the following conditions:

$$|a_{2n+1}| \leqq 1/4, \qquad |a_{2n}| \geqq 25/4 \qquad (n = 1, 2, \cdots). \tag{1.2}$$

Later one of the present authors proved [3] that (1.1) converges if

$$\begin{aligned} |1+a_2| &\geqq 1+|a_1|, \\ |1+a_n+a_{n+1}| &\geqq 1+|a_{n-1}a_n| \quad (n = 2, 3, \cdots). \end{aligned} \tag{1.3}$$

One of the immediate consequences of this theorem is that if

$$\begin{aligned} &|1+a_2| > 1, && |a_2| \geqq (2+m)/(1-m), \\ &|a_{2n+1}| \leqq m < 1, && |a_{2n+2}| \geqq 2+m+m|a_{2n}| \\ &&& (n = 1, 2, 3, \cdots), \end{aligned} \tag{1.4}$$

then (1.1) converges.

Recently Thron [6] has shown that if

$$|a_{2n+1}| \leqq k^2 < 1, \qquad |a_{2n}| \geqq (1+k)^2 + e \qquad (e > 0), \tag{1.5}$$

the continued fraction (1.1) converges. For $k^2<1$ this result can be shown to be a "best" result except possibly for the presence of the quantity e.

The present paper is concerned with establishing convergence criteria of this general type. The principal result is given in Theorem 3.1. The method to be used is the following. Denote the nth approximant of (1.1) by A_n/B_n. Conditions on the numbers a_n are determined which imply that the approximants lie in a given region V of the complex plane. A continued fraction the elements of which are functions of the complex variable z and which reduces to (1.1) for $z=1$ is then introduced. When the given conditions on the numbers a_n are satisfied the approximants of this continued fraction are shown to

Presented to the Society, September 13, 1943, under the title *On convergence regions for continued fractions*; received by the editors June 1, 1943.

[1] Numbers in brackets refer to the Bibliography at the end of the paper.