PROOF OF A THEOREM OF THORIN

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Let

$$A(x, y) = \sum_{j,k=1}^{m,n} a_{jk}x_j y_k$$

be a complex bilinear form, and let $M(\alpha, \beta)$ denote the upper bound of $|A(x, y)|$ for the x's and y's satisfying the condition

$$(\sum |x_j|^{1/\alpha})^\alpha \leqq 1, \qquad (\sum |y_k|^{1/\beta})^\beta \leqq 1.$$

Then $\log M(\alpha, \beta)$ is a convex function of the point (α, β) in the triangle

$$(1) \qquad 0 \leqq \alpha \leqq 1, \qquad 0 \leqq \beta \leqq 1, \qquad \alpha + \beta \geqq 1.$$

The above result is due to M. Riesz.[1] In a recent paper it was shown by Thorin[2] that the theorem is restricted neither to bilinear forms nor to the triangle (1). Thorin's result is as follows.

THEOREM. (i) *Let $f(z_1, z_2, \cdots, z_r)$ be an entire function of r complex variables $z_1, z_2, \cdots, z_r$. Let K be a bounded domain $(v_1, v_2, \cdots, v_r)$ of the r-dimensional Euclidean space, satisfying the conditions $v_1 \geqq 0$, $v_2 \geqq 0, \cdots, v_r \geqq 0$. Let $M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ denote the upper bound of $|f(z_1, z_2, \cdots, z_r)|$ for*

$$|z_1| = v_1^{\alpha_1}, \cdots, |z_r| = v_r^{\alpha_r}, \quad \textit{and} \quad (v_1, v_2, \cdots, v_r) \in K.$$

Then $\log M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ *is a convex function of the point $(\alpha_1, \alpha_2, \cdots, \alpha_r)$ in the domain $0 \leqq \alpha_j < +\infty$, $j = 1, 2, \cdots, r$.*

(ii) *If the points of K satisfy a condition*

$$0 < A \leqq v_j \leqq B < +\infty \qquad (j = 1, 2, \cdots, r)$$

then $\log M(\alpha_1, \alpha_2, \cdots, \alpha_r)$ *is convex in the whole space $-\infty < \alpha_j < +\infty$, $j = 1, 2, \cdots, r$.*

In case (ii), vanishing of some of the α's does not require additional discussion. The situation is slightly different in case (i), since $v_j^{\alpha_j}$ has no meaning if both v_j and α_j are zero. The sets $z_1, z_2, \cdots, z_r$

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[1] M. Riesz, *Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires*, Acta Math. vol. 49 (1926) pp. 465–497.

[2] G. O. Thorin, *An extension of a convexity theorem due to M. Riesz*, Kungl. Fysiografiska Sällskapets i Lund Förhandlinger vol. 8 (1939) no. 14.