ON ORTHOGONAL LATIN SQUARES

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An m-sided Latin square is an arrangement of the numbers $1, 2, \cdots, m$ into m rows and m columns in such a way that no row and no column contains any number twice. Two Latin squares are said to be orthogonal if when one is superimposed upon the other every ordered pair of numbers occurs once in the resulting square. Various methods have been devised for the construction of sets of orthogonal squares. However no method has as yet been given which would yield all possible sets of orthogonal Latin squares. In constructing orthogonal sets it is of value to have simple criteria which enable us to decide whether a given Latin square can be a member of an orthogonal pair.

A Latin square to which an orthogonal square exists will be called a basis square. In this note we shall derive two simple necessary conditions for a square to be a basis square.

THEOREM 1. *If in the Latin square L of side $4n+2$ the square formed by the first $2n+1$ rows and the first $2n+1$ columns contains fewer than $n+1$ numbers which are different from $1, 2, \cdots, 2n+1$ then L is not a basis square.*

PROOF. Denote by I the square formed by the first $2n+1$ rows and the first $2n+1$ columns, by II the square formed by the first $2n+1$ rows and the last $2n+1$ columns, by IV the square formed by the last $2n+1$ rows and the last $2n+1$ columns. Then if a number occurs a times in I it must occur $2n+1-a$ times in II and $2n+1-(2n+1-a)=a$ times in IV. Hence in I and IV together every number occurs $2a$ times. Assume now that I contains fewer than $n+1$ numbers different from $1, 2, \cdots, 2n+1$ and let L' be a square orthogonal to L. In the square resulting from superimposing L' on L every pair $1, i$ $2, i$ $\cdots$ $2n+1, i$ must occur. Hence every number i in L' occurs $2n+1$ times combined with a number of the set $1, 2, \cdots, 2n+1$ in L. But at most $2n$ numbers of the set $1, 2, \cdots, 2n+1$ occur in L outside of I and IV. Hence at least $2n+2$ numbers i of L' occur combined with the numbers $1, 2, \cdots, 2n+1$ in I and IV together an odd number of times. But at most $2n$ numbers of L' occur in I and IV combined with numbers of L which are different from $1, 2, \cdots, 2n+1$.

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