$$\frac{\partial u_p}{\partial r} = \frac{pu_p}{r} + \sum_{k=0}^{n} \beta_k u_{p+k} + \left(\frac{2}{re^{i\theta}}\right) \sum_{k=0}^{m} \sum_{\nu=0}^{m+n} \frac{\alpha_k \gamma_\nu u_{p+k+\nu+1}}{2p + 2\nu + 1}$$

if E has the form II; *and*

$$\frac{\partial u_p}{\partial r} = \frac{pu_p}{r} + \left(\frac{2}{re^{i\theta}}\right) \sum_{k=0}^{m} \sum_{\nu=0}^{n} \left(\frac{\alpha_k \beta_\nu}{2p + 2\nu + 1}\right) u_{p+k+\nu+1}$$
$$+ \gamma_0 u_p + \sum_{k=1}^{m+n} \left(\gamma_k - \frac{2}{re^{i\theta}} \gamma_{k-1}\right) u_{p+k}$$

if E has the form III.

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ON GAUSS' AND TCHEBYCHEFF'S QUADRATURE FORMULAS

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The well known Gauss' Quadrature Formula

$$(1) \qquad \int_{-\infty}^{\infty} G_k(x) d\psi(x) = \sum_{i=1}^{n} \rho_i^{(n)} G_k(\xi_i^{(n)})$$

is valid for every polynomial $G_k(x)$, of degree $k \leqq 2n-1$, the $\{\xi_i^{(n)}\}$ being the roots of the polynomial $P_n(x)$, orthogonal with respect to the distribution $d\psi(x)$ $(i=1, 2, \cdots, n;\ n=1, 2, \cdots)$.[1] If the sequence $\{P_n(x)\}$ is that of Tchebycheff (trigonometric) polynomials, then the Christoffel numbers $\rho_i^{(n)}$, $i=1, 2, \cdots, n$, are equal, and the two quadrature formulas of Gauss and Tchebycheff coincide:

$$(2) \qquad \int_{-\infty}^{\infty} G_k(x) d\psi(x) = \rho_n \sum_{i=1}^{n} G_k(\xi_i^{(n)}), \qquad k \leqq 2n - 1;\ n = 1, 2, \cdots.$$

The converse—that this is the only case of coincidence of these formulas—was proved by R. P. Bailey [1a] and, under more restrictive conditions, by Krawtchouk [1b] (cf. also [2]).[2]

We shall give here four distinct proofs of this statement, without imposing any restrictions on $\psi(x)$.

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[1] $\psi(x)$ is a bounded non-decreasing function, with infinitely many points of increase, for which all moments exist: $c_n = \int_{-\infty}^{\infty} x^n d\psi(x)$; $n=0, 1, 2, \cdots$.

[2] Numbers in brackets refer to the bibliography at the end of the paper.