REMARKS ON TRANSITIVITIES OF BETWEENNESS

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This note provides lattice theoretic interpretations of the transitivities

T_8. $abc \cdot dab \cdot xcd \cdot a \neq b \rightarrow acx$,
T_9. $abc \cdot dab \cdot xcd \cdot a \neq b \rightarrow bcx$,
T_{10}. $abc \cdot abd \cdot xbc \cdot a \neq b \cdot b \neq c \rightarrow xbd$,

introduced by Pitcher and Smiley.[1] It may be recalled that in a lattice the relation abc (b is between a and c) is said to hold if and only if

$$(a \cup b) \cap (b \cup c) = b = (a \cap b) \cup (b \cap c).$$

THEOREM 1. *If L is a lattice then its betweenness relation has one of the transitivities T_8 or T_9 if and only if L is linearly ordered.*

PROOF. It is obvious that T_8 and T_9 are satisfied if L is linearly ordered. To show that T_8 implies linear order, consider two elements $a, c \in L$. Suppose that a and c are not comparable, that is, that none of the relations $a=c$, $a<c$, $a>c$ holds. Then $a \neq a \cup c$, $c \neq a \cup c$. Moreover, we have

$$a\ a \cup c\ c \cdot a \cap c\ a\ a \cup c \cdot a \cup c\ c\ a \cap c \cdot a \neq a \cup c$$

and by T_8 this implies $a\ c\ a \cup c$ which, with $a\ a \cup c\ c$, implies $c = a \cup c$, contrary to our assumption that $c \neq a \cup c$. In the same way T_9 can be shown to imply linear order.

THEOREM 2. *If L is a lattice then its betweenness relation has the transitivity T_{10} if and only if L is linearly ordered or is composed of two linearly ordered systems with a common greatest element, I, and a common least element,* 0.

PROOF. It is easy to see that lattice betweenness in such a lattice has the transitivity T_{10}. Denote the two linearly ordered systems by L_1 and L_2. Then if, in the hypotheses of T_{10}, $b \neq 0$, $b \neq I$, $b \in L_1$, all the elements a, c, d, and x must belong to L_1 and the conclusion follows from the fact that T_{10} holds for linear order. If $b=0$ or $b=I$ in the hypotheses of T_{10} and if $a \in L_1$, then we must have $c \in L_2$, $d \in L_2$, $x \in L_1$ and the conclusion again follows.

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[1] Everett Pitcher and M. F. Smiley, *Transitivities of betweenness*, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 95–114. We shall use the notations and terminology of Pitcher and Smiley.