LAMBERT SUMMABILITY OF ORTHOGONAL SERIES

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If we define Lambert summability of a series, $\sum_1^\infty a_n$, in terms of the existence of the limit

$$(1) \qquad L(a_n) = \lim_{x\to 1-0} (1-x)\sum_1^\infty \frac{na_nx^n}{1-x^n}$$

we have, by a well known theorem of Hardy-Littlewood [1],[1] that $C(a_n)\to L(a_n)\to A(a_n)$; $C(a_n)$, $A(a_n)$ are respectively the Cesàro and Abel means of the series $\sum_1^\infty a_n$.

The proof of $C(a_n)\to L(a_n)$ is elementary in nature, but the proof of $L(a_n)\to A(a_n)$ requires the prime number theorem, and conversely the theorem $L(a_n)\to A(a_n)$ implies the prime number theorem.

For that reason, it is perhaps interesting to show that for orthogonal series of functions $f(x)$, belonging to L^2, the inclusion of $L(a_n)$ between $C(a_n)$ and $A(a_n)$ follows in completely elementary fashion.

That $C(a_n)\sim A(a_n)$ for orthogonal series of L^2 is a known result of Kaczmarz [2]. Hence it is sufficient to show that $L(a_n)\to C(a_n)$. In addition, it is further known that $C(a_n)$ is equivalent to the convergence of the partial sums of the orthogonal series $s_{2^n}(\theta)=\sum_1^{2^n} a_k\phi_k(\theta)$ [3] Therefore, finally, it comes to showing that Lambert summability implies the convergence of the partial sums $s_{2^n}(\theta)$, in order to prove the theorem.

Let $f(\theta)\subset L^2(a, b)$, $a_n=\int_a^b f(\theta)\phi_n(\theta)d\theta$; where $(\phi_n(\theta))$ is an orthonormal sequence in (a, b), $s_n(\theta)=\sum_1^n a_n\phi_n(\theta)$.

Write, where x is $1-1/2^n$,

$$(2) \qquad U_n(\theta) = \sum_1^\infty ka_k\phi_k(\theta)\frac{(1-x)x^k}{1-x^k} - s_{2^n}(\theta) = T_n(\theta) + V_n(\theta)$$

where

$$(3) \qquad T_n(\theta) = \sum_1^{2^n} a_k\phi_k(\theta)\left(\frac{k(1-x)x^k}{1-x^k} - 1\right),$$

$$(4) \qquad V_n(\theta) = \sum_{2^n+1}^\infty ka_k\phi_k(\theta)\frac{(1-x)x^k}{1-x^k}\,.$$

If $\lim_{n\to\infty} U_n(\theta)=0$, the result is proven. To that end, consider the

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[1] Numbers in brackets refer to the references listed at the end of the paper.