CONVERGENCE AND SUMMABILITY PROPERTIES OF SUBSEQUENCES

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In this paper we shall discuss the relation of the convergence or $(C, 1)$ summability of a sequence to that of its subsequences. Some analogous questions for subseries have been considered [6].[2]

Let $\{s_n\}$ be an arbitrary sequence. We can obtain a 1-1 map of its infinite subsequences on the interval $0<t\leqq 1$ as follows. Let $t=.a_1a_2a_3\cdots$ be the infinite dyadic expansion of a point t of the interval. Corresponding to this point we select the following subsequence: retain s_m if the mth place in the expansion, a_m, is 1, and drop it otherwise. The inverse correspondence is evident.

In terms of the Lebesgue measure of sets of points on $(0, 1)$ we are now clearly in a position to speak of "almost all" or "almost none" of the subsequences of $\{s_n\}$.

The problem we have set ourselves is to determine under what conditions does the convergence or summability of a sequence carry over to that of its subsequences, and conversely, whether these properties for suitable subsequences imply them for the sequence itself. For simplicity, we restrict ourselves to sequences of real numbers; although it is apparent that the results are more generally true.

In the case of convergence, the answer to our problem is simple: *a sequence is convergent if and only if almost all of its subsequences are convergent.* In the case of $(C, 1)$ summability, the problem is more difficult. It has been established that *all* the subsequences of a sequence cannot be summable by a fixed regular matrix method unless the sequence is in fact convergent [2]. In §§3, 4 we discuss, for $(C, 1)$ summability, the consequences of replacing *all* by *almost all.* We show, for example, that $\{s_n\}$ is $(C, 1)$ summable if almost all of its subsequences are, but not conversely.

The principal tools are the Rademacher functions, $R_n(t)$, and the properties of homogeneous sets.

Our results have certain obvious connections with probability; these are discussed in §5.

1. **Preliminary results.**

All sets which occur in the sequel are to be taken as subsets of $(0, 1)$; a.e. will mean "almost everywhere in $(0, 1)$."

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[2] Numbers in brackets refer to the references listed at the end of the paper.