A THEOREM ON GENERALIZED DERIVATIVES

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1. **General remarks.** Suppose that a function $f(x)$, defined in the neighborhood of a point x_0, satisfies a relation

$$f(x_0 + t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2/2! + \cdots + \alpha_{k-1}t^{k-1}/(k-1)! + \omega_k(t)t^k/k!, \tag{1.1}$$

where $\alpha_0, \alpha_1, \cdots, \alpha_{k-1}$ are independent of t, and the expression $\omega_k(t) = \omega(x_0, t)$ approaches a finite limit α_k as t tends to 0. The function f is then said to possess a kth generalized derivative at the point x_0, and α_k is the value of that derivative. Instead of α_k we shall write $D_k f(x_0)$. It is clear that the existence of $D_k f(x_0)$ implies that of $D_{k-1}f(x_0)$.

The existence of $D_0 f(x_0)$ is simply continuity of the function f at the point $x = x_0$. For $k=1$ the definition of $D_k f(x_0)$ is equivalent to that of the ordinary derivative $f^{(k)}(x_0)$. No such equivalence exists for higher values of k, for then the existence of $D_k f(x_0)$ does not even imply continuity of f for $x \neq x_0$. However, if $f^{(k)}(x_0)$ exists and is finite, then $D_k f(x_0)$ also exists and is equal to $f^{(k)}(x_0)$.

It is a classical result of Fatou that, if a function $f(x)$ is everywhere continuous and, say, of period 2π, then the integral

$$\int_0^\pi \frac{f(x+t) - f(x-t)}{t}\,dt = \lim_{\delta \to +0} \int_\delta^\pi \frac{f(x+t) - f(x-t)}{t}\,dt$$

exists for almost every x.[1] This integral may also be written

$$\int_0^\pi \frac{\omega_0(x, t) - \omega_0(x, -t)}{t}\,dt \tag{1.2}$$

or

$$\int_0^\pi \frac{\epsilon_0(x, t) - \epsilon_0(x, -t)}{t}\,dt \tag{1.3}$$

if for any x for which $D_k f(x)$ exists we introduce the notation

$$\omega_k(x, t) = D_k f(x) + \epsilon_k(x, t)$$

(so that $\epsilon(x, t)$ tends to 0 with t, for x fixed).

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[1] For this and more general results, see, for example, the author's *Trigonometric Series*, Warsaw, 1935, chap. 7.