# ON SUBHARMONIC FUNCTIONS

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If $p(x, y)$ is continuous in a domain (non-null connected open set) $\mathcal{D}$, then $p(x, y)$ is subharmonic in $\mathcal{D}$ if and only if the inequality

$$p(x, y) \leqq A(p; x, y; r) \equiv \frac{1}{\pi r^2} \iint_{D(x,y;r)} p(x+\xi, y+\eta) d\xi d\eta$$

holds for all circular discs $D(x_0, y_0; r): (x-x_0)^2+(y-y_0)^2=\xi^2+\eta^2 \leqq r^2$ in $\mathcal{D}$. If $p(x, y)$ is continuous along with its partial derivatives of the second order in $\mathcal{D}$, then $p(x, y)$ is subharmonic there if and only if $\Delta p(x, y) \geqq 0$ in $\mathcal{D}$, where $\Delta$ is the Laplace operator [2, 3].[1]

If $p(x, y)$ is continuous in $\mathcal{D}$, then $p(x, y)$ is said to be of class $PL$ [2] in $\mathcal{D}$ provided (i) $p(x, y) \geqq 0$ and (ii) $\log p(x, y)$ is subharmonic wherever $p(x, y) \neq 0$. If $p(x, y) \geqq 0$ and is continuous along with its partial derivatives of the second order, then $p(x, y)$ is of class $PL$ if and only if $p\Delta p - p_x^2 - p_y^2 \geqq 0$ wherever $p(x, y) \neq 0$.

Beckenbach [1] has proved the following theorem characterizing functions of class $PL$.

THEOREM A. *If $p(x, y) \geqq 0$ in $\mathcal{D}$, then $p(x, y)$ is of class $PL$ in $\mathcal{D}$ if and only if $[(x-\alpha)^2+(y-\beta)^2]p(x, y)$ is subharmonic in $\mathcal{D}$ for every choice of the real constants $\alpha$, $\beta$.*

The Beckenbach theorem is comparable to the classic Montel-Radó theorem, which was later generalized by Kierst and Saks [3, 4] for functions $p(x, y)$ with continuous partial derivatives of the second order; this generalization is the following theorem.

THEOREM B. *Let $f(t)$ have a continuous second derivative, with $f'(t) > 0$, for $-\infty < t < \infty$. If $v(x, y)$ has continuous partial derivatives of the second order in $\mathcal{D}$, and if the function $f(\alpha x+\beta y+v(x, y))$ is subharmonic in $\mathcal{D}$ for every choice of the real constants $\alpha$, $\beta$, then $v(x, y)$ is subharmonic in $\mathcal{D}$.*

The question arises as to the possibility of exhibiting a Kierst-Saks type of generalization for Beckenbach's Theorem A. Our result is the following.

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[1] The numbers in square brackets refer to references listed in bibliography at end of paper.