THE TERNARY OPERATION $(abc) = ab^{-1}c$ OF A GROUP

JEREMIAH CERTAINE

1. Introduction. This note is the result of some investigations into the ternary operation $ab^{-1}c$ in a group. We shall assume familiarity on the part of the reader with the notions of a group, a one-one transformation (we shall use the shorter term permutation) of an arbitrary set of elements, an automorphism, and a coset.[1] We shall use the multiplicative notation for a group G with elements $a, b, c, \cdots$. We shall also use the following convention for multiplication of permutations. Given two permutations T_i: $x \rightarrow xT_i$ $(i=1, 2)$, then T_1T_2 is $x \rightarrow (xT_1)T_2$. Finally, we denote automorphisms by small Greek letters.

In §2 we shall review certain properties of the ternary operation in a given group, determining all subsets closed with respect to this operation and the group of permutations of G which preserve this operation. These results had been previously obtained by Reinhold Baer.[2]

In §§3 and 4 we give postulates for this operation with proofs of their independence and consistency. Thus, if a ternary operation satisfies these postulates in an arbitrary set of elements, then the set may be made into a group (unique within isomorphism) in which $(abc) = ab^{-1}c$. The first set of postulates appears as a weakened form of a set given by Baer in his paper,[3] in which he mentions the group property. This and an equivalent set completely determine the ternary function as $ab^{-1}c$. However, by further weakening one of these postulates, it is possible to get a system which no longer has this last property. That is, the group property still holds but the ternary operation is not determined by the group operation.

In the remaining sections we get a geometric interpretation of the ternary operation and derive therefrom simple conditions on pairs of elements (vectors) under which they form a group. In the case where an abelian group is desired, the conditions are even simpler, reducing essentially to a single law.

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[1] Cf. H. Zassenhaus, *Lehrbuch der Gruppentheorie*, Leipzig and Berlin, 1937, pp. 1, 5, 41 and 10.

[2] *Zur Einführung des Scharbegriffs*, J. Reine Angew. Math. vol. 160 (1929) pp. 199–206.

[3] Cf. Baer, op. cit. p. 202, footnote.