SUBSERIES OF A CONVERGENT SERIES

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In a recent paper J. D. Hill[1] has discussed the mean-value of the subseries of any absolutely convergent series $s=\sum u_n$. Simplifying his method by use of the Rademacher functions, we obtain a mapping of the subseries into the interval $0\leqq x\leqq 1$ by defining[2]

$$\phi(x) = \sum_{n=1}^{\infty} \frac{1+R_n(x)}{2} u_n. \tag{1}$$

Hill's result states that if $\sum |u_n|$ converges, then the mean-value is given by

$$\int_0^1 \phi(x)dx = s/2. \tag{2}$$

In the theorem below we point out the weakest condition on the series $\sum u_n$ for which this result persists.

LEMMA. *If* (1) *converges on a set of positive measure it converges almost everywhere.*

Let D be the set of points on which (1) converges. Let $x=a_1a_2a_3\cdots$ (in binary notation) be a point of D. If a finite number of the a_i are changed then the new point still belongs to D, for by the definition of the Rademacher functions this operation changes only a finite number of the terms of the series (1). Then D is a "homogeneous" set not of measure 0; hence it must be of measure 1.[3]

THEOREM. *A necessary and sufficient condition that the series* (1) *converge on a set of positive measure is that the two series* $\sum u_n$ *and* $\sum u_n^2$ *converge. Then* (1) *converges almost everywhere and* (2) *is valid.*

(i) Suppose that (1) converges on a set of positive measure. Then it must, by the lemma, converge almost everywhere. Then there exist

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[1] Bull. Amer. Math. Soc. vol. 48 (1942) p. 103.

[2] The mapping is not 1-1 at the points $x=k/2^n$, but this does not affect the results. For the properties of the Rademacher functions used in this paper see Kacmarcz and Steinhaus, *Le système orthogonale de M. Rademacher*, Studia Mathematica vol. 2 (1939) p. 231.

[3] C. Visser, *The law of nought-or-one in the theory of probability*, Studia Mathematica vol. 7 (1938) pp. 146–147.