Further

$$v(r, \theta) = \frac{1}{\pi}\int_{-\pi}^{\pi} K(r, \theta - x)V(1, x)dx = \frac{1}{\pi}\int_{\alpha}^{\beta} K(r, \theta - x)V(1, x)dx$$

is clearly a function, harmonic in $r<1$ and it is easy to deduce from the properties of $K(r, \theta)$ and $V(1, \theta)$ that $v(r, \theta)$ satisfies all the conditions of the theorem.

COROLLARY. *If $f(z)$ is analytic in $|z|<1$, then, given the arc ($|z|=1$, $\alpha<\arg z<\beta$), there is a function $g(z)$ analytic in $|z|<1$ and on the arc of $|z|=1$ complementary to (α, β), such that $f(z)-g(z)$ can be extended analytically across (α, β).*

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ON THE COMPLEX ZEROS OF THE BESSEL FUNCTIONS

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1. **Introduction.** Various proofs have been given for the following classical theorem of A. Hurwitz:

THEOREM 1. *The entire function*

$$(1.1) \qquad z^{\beta/2}J_{-\beta}(2z^{1/2}) = \sum_{m=0}^{\infty} \frac{(-z)^m}{m!}\frac{1}{\Gamma(m+1-\beta)}$$

has precisely $[\beta]$ nonpositive zeros. Here $J_{-\beta}$ is the Bessel function of order $-\beta$ and $\beta \geqq 0$.

In case β is an integer these nonpositive zeros are all at the origin; in case β is not an integer and $[\beta]$ is odd there is precisely one negative zero and we have $\frac{1}{2}([\beta]-1)$ pairs of conjugate complex zeros; in case β is not an integer and $[\beta]$ is even there are $\frac{1}{2}[\beta]$ pairs of conjugate complex zeros.

Most of the proofs for this theorem (see the papers [**2**, **4**, **6**, **7**, **9**] of the Bibliography at the end of the text) make use of polynomial approximations of the Bessel function. The present proof follows the same line by obtaining the Bessel function as the limit of *Laguerre polynomials*. The study of the complex zeros of these polynomials is

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