SOME SIMPLE DIFFERENTIAL DIFFERENCE EQUATIONS AND THE RELATED FUNCTIONS

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1. **Introduction.** The work of Brook Taylor, Jean Bernoulli and Daniel Bernoulli led to the formulation of differential difference equations which are all included in the equation

$$(1.1)\quad \begin{aligned} &Md^2u_n/dt^2 + 2Kdu_n/dt + Su_n \\ &\qquad = (na + a + b)(u_{n+1} - u_n) - (na + c)(u_n - u_{n-1}) \end{aligned}$$

in which M, K, S, a, b, c are constants. This equation may be treated in at least six different ways, which may be described briefly as follows:

(1) The method of simple solutions in which the aim is to express the desired solution as a sum of simple solutions of type

$$(1.2)\qquad u_n(t) = U_n(p) \exp (ipt),$$

in which p is a constant which may assume a certain set of values. This method was used with great success by the writers just named and was much improved by L. Euler, J. le Rond d'Alembert, J. L. Lagrange and J. Fourier. The method has been greatly developed during the last one hundred fifty years. Some idea of this development may be derived from the excellent report of H. Burkhardt[1] on expansions in series of oscillating functions.

(2) The method of generating functions in which a differential equation is formed for the generating function

$$G(z, t) = \sum z^n u_n(t).$$

This method may, perhaps, be associated with the names of Lagrange and Laplace as these writers developed a theory of generating functions. The important developments for the differential difference equation came quite late and began, perhaps, with the work of Koppe[2] on the function which I shall call the *influence function* for

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[1] H. Burkhardt, *Entwicklungen nach oscillirenden Funktionen und Integration der Differentialgleichungen der mathematischen Physik*, Jber. Deutschen Math. Verein. vol. 10 (1908) pp. 1–804.

[2] M. Koppe, *Die Ausbreitung einer Erschütterung an der Wellenmaschine durch einen neuen Grenzfall der Besselschen Functionen*, Program Andreas-Real-gymnasium, Berlin, No. 96 (1901) 28 pp. See also T. H. Havelock, *On the Instantaneous propagation of disturbance in a dispersive medium*, Philosophical Magazine (6) vol. 19 (1910) pp. 160–168; E. Schrödinger, *Dynamik elastischer gekoppelte Punktsysteme*, Annalen