SOME THEOREMS ON THE EULER ϕ-FUNCTION

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The Euler ϕ-function, $\phi(m)$, denotes the number of positive integers not greater than m which are relatively prime to m.[1] It was noted by U. Scarpis[2] that $n|\phi(p^n-1)$. Generalizations of this result are obtained in Theorems 9 and 10.

The first five theorems are either well known or self-evident.[3]

THEOREM 1. *If $p_1, \cdots, p_k$ are the distinct prime factors of m, then*

$$\phi(m) = m(p_1 - 1)(p_2 - 1) \cdots (p_k - 1)/p_1p_2 \cdots p_k.$$

THEOREM 2. *If $a_1, \cdots, a_k$ are relatively prime in pairs, then*

$$\phi(a_1 \cdots a_k) = \phi(a_1)\cdot\phi(a_2) \cdots \phi(a_k).$$

THEOREM 3. *If w is the product of the distinct prime factors common to m and n, then*

$$\phi(mn) = w\cdot\phi(m)\cdot\phi(n)/\phi(w).$$

THEOREM 4. *If $a|b$, then $\phi(a)|\phi(b)$.*

THEOREM 5. *If $q|a$ and $q\equiv 1 \pmod{p^\alpha}$, then $p^\alpha|\phi(a)$.*

THEOREM 6. *If p is an odd prime, $a\neq b$, and $\alpha \geqq 1$, then*

$$p^{2\alpha-1} \mid \phi(a^{p^\alpha} + b^{p^\alpha}).$$

The proof is by induction on α. We assume $a>b$. In the notation of Birkhoff and Vandiver,[4] $a^p+b^p=V_{2p}/V_p$. By their Theorems V and I, there is a prime divisor q of a^p+b^p such that $q\equiv 1 \pmod p$ unless $p=3$, $a=2$, $b=1$. Then by Theorem 5, $p|\phi(a^p+b^p)$, and in the exceptional case, $3|\phi(2^3+1^3)$. Thus the theorem holds for $\alpha=1$, starting the induction, so we assume it for all positive integers less than α. We adopt the notation $C=AB$, where

$$C = a^{p^\alpha} + b^{p^\alpha}, \qquad P = p^{\alpha-1}, \qquad A = a^p + b^p,$$

$$B = a^{(p-1)P} - a^{(p-2)P}\cdot b^P + \cdots - a^P\cdot b^{(p-2)P} + b^{(p-1)P}.$$

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[1] In this discussion all letters represent positive integers. In particular, p and q represent primes.

[2] Period. Mat. vol. 29 (1913) p. 138.

[3] See, for example, L. E. Dickson, *History of the theory of numbers*, vol. 1, chap. 5.

[4] Ann. of Math. (2) vol. 5 (1903) pp. 173–180.