ON THE ZEROS OF THE DERIVATIVES OF A FUNCTION AND ITS ANALYTIC CHARACTER

G. POLYA

1. Introduction. How do the zeros of the nth derivative $f^{(n)}(x)$ behave when n becomes very large? How does this behavior depend on the analytic nature of the function $f(x)$? At first sight this question appears to be a little out of the way. In fact it is intimately connected with essential parts of the theory of functions. In the Hadamard theory of the singularities of the Taylor series we consider the sequence of the derivatives $f(x), f'(x), f''(x), \cdots$ for a fixed complex value of x; in the theory of quasi-analytic functions we consider the same sequence for variable real x, and we consider especially the maximum absolute value of each of its terms in a given interval; now we consider again the same sequence $f(x)$, $f'(x)$, $f''(x)$, $\cdots$ for variable x, that may be complex or real, and we consider especially the values of x for which its terms vanish. In all three cases the main object of consideration is the connection of the chosen feature of the sequence $f(x), f'(x), f''(x), \cdots$ with the analytic nature of the function $f(x)$.

There is a special reason for giving a general survey of results about the zeros of successive derivatives just now. The subject received quite recently essential contributions from several mathematicians in this country, from R. P. Boas, Einar Hille, A. C. Schaeffer, I. J. Schoenberg, Gabor Szegö, J. D. Tamarkin, D. V. Widder, Norbert Wiener and the present speaker. All these contributions are linked together and seem to open the door to further results and to a well connected harmonious theory.

In this short talk I should like to emphasize the main outline and the essential divisions of the theory in so far as they are recognizable today. In order to do this as clearly and intuitively as I can, I have to discuss some of the older results, to leave out some of the newer ones and to sacrifice precise details except for a few central points.

Our subject has two main parts. In one we consider analytic functions of a complex variable, in the other real-valued functions of a real variable. I proceed to describe these two parts in this order.

An address presented before the New York meeting of the Society on April 3, 1942, by invitation of the Program Committee; received by the editors May 8, 1942.