# RESIDUE THEOREMS OF HARMONIC FUNCTIONS OF THREE VARIABLES

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1. **Introduction.** The connection between analytic functions of one complex variable and harmonic functions of two real variables is used in the study of the latter functions. The analogy suggests generalizations of these methods of attack to the case of three variables.

Suppose that $O$ is some operation which transforms the class of analytic functions of the complex variable $u$ into the class of harmonic functions of three variables $x_1, x_2, x_3$. If $O$ possesses the property that to every harmonic polynomial of the $n$th degree there corresponds an expression of the form $A\ [u(x_1, x_2, x_3)]^n$, where $u$ is a linear expression in $x_1, x_2, x_3$, then such an operation yields a representation of spherical harmonics.

The existing representations of the spherical harmonics lead us to several operations $O$. The operation P described below (a generalization of the known Whittaker's formula) seems to be the most suitable for the questions considered in this note.

Let $F$ be the class of analytic functions of complex variables $u$ and $\zeta$, where $u = x_1 + (i/2)(x_2 + ix_3)\zeta^{-1} + (i/2)(x_2 - ix_3)\zeta$. By the operation

$$(1.1)\quad \mathrm{P}(f, \mathfrak{L}^1, \mathfrak{T}) = (1/2\pi i)\int_{\mathfrak{L}^1} f(u, \zeta)d\zeta, \qquad \mathfrak{X} = (x_1, x_2, x_3) \in \mathcal{V}^3(\mathfrak{T})$$

$F$ is transformed into the class of (complex) harmonic[1] functions (see [1, 2][2]). $\mathcal{V}^3(\mathfrak{T})$ denotes a sufficiently small neighborhood of the point $\mathfrak{T} = (t_1, t_2, t_3)$, and $\mathfrak{L}^1$ is a closed curve in the $\zeta$-plane.

$\mathrm{P}(f, \mathfrak{L}^1, \mathfrak{T})$ is a functional. If we carry out the integration for $\mathfrak{X} \in \mathcal{V}^3(\mathfrak{T})$ and we continue analytically the obtained harmonic function $H(\mathfrak{X})$ to a point $\mathfrak{Y} \in \mathcal{V}^3(\mathfrak{B})$, then the continued function is not necessarily equal to $\mathrm{P}(f, \mathfrak{L}^1, \mathfrak{B})$. This is the reason why it is necessary to indicate the point $\mathfrak{T}$ in the neighborhood of which the integration is carried out.

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[1] Here and hereafter harmonic functions mean harmonic functions of three *real* variables $x_1, x_2, x_3$, except when we specifically state that we mean harmonic functions of two variables. By taking the real or imaginary part in the formulas for complex harmonic functions we obtain analogous results for real harmonic functions.

[2] The numbers in brackets refer to the bibliography.

E [ $\cdots$ ] means the set of points satisfying the relations indicated in the brackets. Superscripts over a manifold indicate its dimensionality.