ON AN ELEMENTARY ANALOGUE OF THE RIEMANN-MANGOLDT FORMULA

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Ramanujan's unsuccessful approach to the Prime Number Theorem, published only recently, is based on the power series

$$\sum_{n=1}^{\infty} \Lambda(n) x^n \equiv \sum_p \log p \sum_{k=1}^{\infty} x^{p^k} \quad \text{and} \quad \sum_{n=1}^{\infty} p^n x^{p^n}, \qquad 0 < x < 1,$$

where p denotes a prime and p is 2 in the last series. In his discussion of Ramanujan's failure in case of the latter series, a series impracticable as $x \to 1$, Hardy gives for the function represented by the series another expansion, one exhibiting the critical "wobbles," as follows:[1]

$$\sum_{n=1}^{\infty} p^n \exp(-p^n s) = \{\quad\}/\log p, \qquad p = 2, \tag{1}$$

where $\{\quad\}$ is the expression

$$\begin{aligned} \{\quad\} = \frac{1}{s} - \frac{1}{s} \cdot \sum_{k=-\infty}^{\infty}{}' \Gamma\left(\frac{1 + 2\pi k i}{\log p}\right) s^{-2\pi k i/\log p} \\ - \log p \sum_{n=0}^{\infty} \frac{(-1)^n p^{n+1}}{p^{n+1} - 1} \frac{s^n}{n!}; \end{aligned} \tag{2}$$

it being understood that $\sum_{k=-\infty}^{\infty \prime} = \sum_{k=-\infty}^{-1} + \sum_{k=1}^{\infty}$ and $-\log x = s > 0$.

It will be seen later on that Hardy's result (2) contains two errors. However, the purpose of this note is not calculation of the corrections necessary, which are of a trivial nature, but the presentation of a short approach which seems to be of methodical and historical interest.

First, (2) is of the same type as the "explicit formula" of Riemann-Mangoldt[2] (the two sums representing the contributions of the "non-trivial" and "trivial" zeros, respectively). Correspondingly, Hardy's

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[1] G. H. Hardy, *Ramanujan*, Cambridge, 1940, chap. II, formulae (2.9.1) and (2.11.2).

[2] Cf., for example, A. E. Ingham, *The Distribution of Primes*, Cambridge Tracts, no. 30 (1932), chap. IV.

Relevant for the comparison is only the "Abelian" form (instead of the deeper "Cesàro" form) of the Riemann-Mangoldt formula; cf. G. H. Hardy and J. E. Littlewood, Acta Mathematica, vol. 41 (1918), pp. 119–196.