ON SOME PROPERTIES OF SYMMETRICAL PERFECT SETS[1]

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This paper deals with some properties of symmetrical perfect sets, in view of applications: (i) to the problem of multiplicity of trigonometrical series; (ii) to the study of Fourier-Stieltjes coefficients of singular monotone functions; (iii) to the problem of absolute convergence of trigonometrical series.

1. **Sets of multiplicity of trigonometrical series.** We shall consider, throughout the paper, symmetrical perfect sets, that is, sets which are obtained, in the closed interval $(0, 2\pi)$ by the following process. We divide the interval in three parts of lengths proportional to ξ_1, $1-2\xi_1$, ξ_1, and we remove the central open interval. In the second operation we divide each one of the two intervals left in three parts proportional to ξ_2, $1-2\xi_2$, ξ_2, and we remove the two central open intervals, and so on infinitely, the sequence $\{\xi_p\}$ being such that $0<\xi_p\leqq \frac{1}{2}$.[2] After p operations, we have thus removed 2^p-1 intervals which we shall denote by δ_{pk} $(k=1, 2, \cdots, 2^p-1)$ and 2^p intervals are left, which we shall denote by η_{pk} $(k=1, 2, \cdots, 2^p)$. Each interval η_{pk} is of length equal to $\eta_p=2\pi\cdot\xi_1\xi_2\cdots\xi_p$. Let E_p be the set constituted by the 2^p intervals η_{pk} and let $E(p)$ be its measure. We have

$$E(p) = 2^p\eta_p = 2\pi\cdot 2^p\xi_1\xi_2\cdots\xi_p. \tag{1}$$

$E(p)$ is a non-increasing function of p and the measure of the perfect set P obtained by the above-described process is lim $E(p)$ for $p=\infty$. We shall only consider, throughout the paper, sets for which this limit is equal to zero.

We shall now construct a monotone continuous function $F(x)$ constant in every interval contiguous to P but increasing from one interval to another, by the following well known process.[3] For every p let F_p be a non-decreasing continuous function defined by the following conditions: $F_p(0)=0$, $F_p(2\pi)=1$,

$$F_p(x) = \frac{k}{2^p} \text{ in } \delta_{pk}, \qquad k = 1, 2, \cdots, 2^p - 1,$$

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[2] If $\xi_p=\frac{1}{2}$ no interval is removed in the pth operation but the intervals left after the $(p-1)$th operation are subdivided in two equal parts, and we deal in the $(p+1)$th operation, with the intervals thus subdivided.

[3] See Menchoff [1]; Zygmund [1, p. 294].