MEAN-VALUE SURFACES

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Introduction. The real functions

$$x_j = x_j(u, v), \qquad j = 1, 2, 3, \tag{1}$$

defined and continuous in a finite simply connected domain[1] D, will be said to define a surface S. If the first partial derivatives of the functions (1) are continuous in D, and if

$$E(u, v) = G(u, v), \qquad F(u, v) = 0 \tag{2}$$

hold in D, where

$$E(u, v) \equiv \sum_{j=1}^{3} x_{ju}^2, \qquad F(u, v) \equiv \sum_{j=1}^{3} x_{ju}x_{jv}, \qquad G(u, v) \equiv \sum_{j=1}^{3} x_{jv}^2$$

are the coefficients of the first fundamental quadratic form of S, then the surface is said to be given in isothermic representation by the functions (1) and the parameters u, v are said to be isothermic parameters; the map of D on S is conformal except where $E=G=0$.

In a previous paper,[2] the authors studied the equation

$$\sum_{j=1}^{3}\left[\int_C x_j(u, v)dz\right]^2 = 0, \qquad z = u + iv, \tag{3}$$

where C is a circle in D; the following necessary and sufficient condition was obtained.

THEOREM A. *If the functions* (1) *have continuous partial derivatives of the third order in a finite simply connected domain D, then a necessary and sufficient condition that they map D isothermically either on a surface S that lies on a sphere of finite non-null radius, such that circles are mapped on circles, or on a minimal surface S, is that* (3) *hold for each circle C in D.*

1. **Mean-value surfaces.** Let the coordinate functions (1) of a surface S be continuous in a finite simply connected domain D; then the circular averages

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[1] A domain is a non-null connected open set.

[2] *Generalizations to space of the Cauchy and Morera theorems*, Transactions of this Society, vol. 49 (1941), pp. 354–377; in particular, see p. 365.