CONTINUA OF MINIMUM CAPACITY[1]

G. C. EVANS

1. **Surfaces containing a given volume.** In an endeavour to simplify a proof of Liapounoff [2], to the effect that in the problem of the forms of equilibrium of rotating liquids the sphere would be the only form for a liquid at rest, Poincaré [1] was led to the consideration of electric capacities of solids of given volume, and arrived at the result that among such bodies the sphere would have minimum capacity. The present paper originated in the question of the determination of the surface sheet, without volume, which would be bounded by a given closed curve in space, and, among all such surfaces, have minimum capacity.

In the discussion of his problem, Poincaré assumes tacitly that there do exist one or more bodies of the given volume, with smooth boundaries, which furnish relative minima for the capacity with respect to neighboring forms; and his treatment amounts to a proof that among these the sphere furnishes the absolute minimum.

Let the body F, with smooth exterior boundary S, be considered as a conductor on which a positive charge m is spread so as to be in equilibrium—that is, the charge lies entirely on the surface S with a surface density $\sigma(P)$, and its potential $V(M)$ has a constant value V_0 within S, is continuous across S, satisfies Laplace's equation $\nabla^2 V=0$ outside S and vanishes at infinity. The density on S is given by the equation

$$\sigma(P) = -\frac{1}{4\pi}\frac{dV}{dn}, \qquad n \text{ the exterior normal.}$$

The energy of the distribution may be written in the equivalent forms

$$(1)\quad I = \frac{1}{8\pi}\int_W (\nabla V)^2 dP = \frac{1}{8\pi}\int_W (V_x^2 + V_y^2 + V_z^2)dP,\ W = \text{entire space},$$

$$(2)\quad I = \frac{1}{2}\int_S \sigma(P)V_0 dP = \tfrac{1}{2}mV_0.$$

The capacity of K of F is defined by the equation

$$(3)\qquad KV_0 = m \quad\text{or}\quad I = \frac{m^2}{2K} = \tfrac{1}{2}KV_0^2.$$

[1] Presidential address presented to the Society, January 1, 1941, under the title *Surfaces of minimum capacity.*