THE CONSTRUCTION OF POSITIVE TERNARY QUADRATIC FORMS[1]

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1. **Binary quadratic forms.** A positive, binary quadratic form $f=(a, t, b)=ax^2+2txy+by^2$ (a, t, and b real numbers) is equivalent to one and only one form in which

$$(1) \qquad |2t| \leqq a \leqq b, \quad t \geqq 0 \text{ if } a = b \text{ or } a = |2t|.$$

Such a form is called *reduced*, and has the further properties that: (i) a is the least number properly represented by f; (ii) $ab \leqq 4\Delta/3$, where $\Delta = ab - t^2$; (iii) among all forms equivalent to f with the minimum a as first coefficient, b is the least possible.

To obtain all reduced, classical, binary quadratic forms of a given determinant Δ, we have the following well known algorithm: factor $\Delta+t^2$ ($\pm t=0, 1, \cdots, (\Delta/3)^{1/2}$) as ab, with $|2t| \leqq a \leqq b$, in all possible ways; but discard forms with $t<0$ if $a=b$ or $|2t|$.

2. **Ternaries.** We develop in this article a similar method of finding a unique form in every class of integral ternary quadratic forms of a *given* determinant, or in a given order or genus. The methods of reduction hitherto devised (Seeber [**1**], Eisenstein [**2**], Selling [**3**]) are entirely adequate if one wishes to calculate all reduced forms of determinant less than a certain fixed value, but are not connected closely enough with the invariants to make the computation of forms with given values for their invariants practicable. By the method of this article one can obtain the reduced forms in a given genus of determinant around 1000 in fifteen minutes.

Our reduced form is not the same as that of Eisenstein or Selling, but we shall see how to pass from our form to that of Eisenstein.

Eisenstein found that within every class of real, positive, ternary quadratic forms $f=(a, b, c, r, s, t)=ax^2+by^2+cz^2+2ryz+2sxz+2txy$ there is a unique form (to be called *E-reduced*) satisfying the following inequalities:

$$(2) \qquad r, s, t \text{ all} > 0, \text{ or all} \leqq 0;$$

$$(3) \qquad a \geqq 2|s|, \quad a \geqq 2|t|, \quad b \geqq 2|r|;$$

$$(4) \qquad a \leqq b \leqq c; \quad a+b+2r+2s+2t \geqq 0;$$

$$(5) \qquad \text{if } a = b, \ |r| \leqq |s|; \text{ if } b = c, \ |s| \leqq |t|;$$

[1] Presented to the Society, May 2, 1941.