# RATIONAL APPROXIMATIONS TO IRRATIONALS

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It is well known that if $p/q$ is a convergent to the irrational number $x$, then $|x-p/q|<1/q^2$. The immediate converse is of course false but I have not seen in the literature[1] any statement of the converse which is given below.

THEOREM 1. *If $p$ and $q$ are coprime, $q>0$, and if $|x-p/q|<1/q^2$, then necessarily $p/q$ is one of the three (irreducible) fractions*

$$p'/q', \qquad (p'+p'')/(q'+q''), \qquad (p'-p'')/(q'-q''),$$

*where $p''/q''$, $p'/q'$ are two consecutive convergents to the irrational $x$. One at least of the two fractions $(p'+\epsilon p'')/(q'+\epsilon q'')$ where $\epsilon=\pm 1$ satisfies the inequality.*

In other words if the inequality is satisfied, then

$$p/q = [a_1, a_2, \cdots, a_{n-1}, a_n + c], \qquad c = 0, \pm 1,$$

where $[a_1, a_2, \cdots, a_r, \cdots]=x$ is the infinite simple continued fraction for $x$, so that the $a_i$ are integers, $a_i \geqq 1$ $(i\geqq 2)$.

Suppose that $x-p/q=\epsilon\theta/q^2$, $0<\theta<1$, $\epsilon=\pm 1$. Let

$$p/q = [b_1, b_2, \cdots, b_m], \qquad p'/q' = [b_1, b_2, \cdots, b_{m-1}],$$

where $m$ (which we can choose to be odd or even) is taken so that $(-1)^{m-1}=\epsilon$. Defining $y$ by the equation

$$x = [b_1, b_2, \cdots, b_m, y] = (yp + p')/(yq + q'),$$

we obtain $\epsilon\theta=q^2(x-p/q)=(p'q-pq')q/(yq+q')$; so that, since $p'q-pq'=(-1)^{m-1}=\epsilon$, $y+q'/q=1/\theta$.

Since $1/\theta>1$ and $q'/q<1$ it follows that $y>0$.

If $y>1$, then $y=[b_{m+1}, b_{m+2}, \cdots]$ $(b_{m+1}\geqq 1, \cdots)$, and so $x=[b_1, b_2, \cdots, b_m, b_{m+1}, \cdots]$, which, since the infinite simple continued fraction is unique, shows that $p/q=[b_1, \cdots, b_m]$ is the $m$th convergent to $x$. If however $y<1$, then $1/y=[c, b_{m+1}, b_{m+2}, \cdots]$ with $c\geqq 1$. But $q/q'=[b_m, b_{m-1}, \cdots, b_2]$ and therefore one of $c$ and $b_m$ must be unity for, if not, then $1/y>2$, $q/q'>2$, $y+q'/q<1<1/\theta$.

[1] Editor's note. In the meantime, R. M. Robinson has proved similar results in the Duke Mathematical Journal, vol. 7 (1940), pp. 354–359. Also the first part of Theorem 1 was observed by P. Fatou, Comptes Rendus de l'Académie des Sciences, Paris, vol. 139 (1904), pp. 1019–1021.