SOME MORE UNIFORMLY CONVEX SPACES[1]

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Let $\{B_i, i=1, 2, \cdots\}$ be a sequence of Banach spaces, and define $B=\mathfrak{P}^p\{B_i\}$ to be the space of sequences $b=\{b_i\}$ with $b_i \in B_i$ and $\|b\|=(\sum_i\|b_i\|^p)^{1/p}<\infty$, $1<p<\infty$. It is known that B, normed in this way, is also a Banach space. Boas[2] showed that if $B_i=l^p$ for all i or if $B_i=L^p$ for all i then B is uniformly convex. Since then I have shown[3] that if $B_i=l^{p_i}$ or L^{p_i}, $1<p_i<\infty$, and if the sequence $\{p_i\}$ is not bounded away from 1 and ∞ (that is, if there do not exist $1<m \leqq M<\infty$ with $m \leqq p_i \leqq M$ for all i), then B is not uniformly convex and cannot be renormed to an isomorphic uniformly convex space. One purpose of this note is to fill in the gap between these results by means of the following theorem.

THEOREM 1. *If $B_i=l^{p_i}$ or L^{p_i}, $1<p_i<\infty$, and if the sequence $\{p_i\}$ is bounded away from* 1 *and* ∞, *then $B=\mathfrak{P}^p\{B_i\}$ is uniformly convex.*

Another question which is answered here is raised by Boas (loc. cit., Footnote 3). In Boas' notation $l^p(B_0)$ is the space $\mathfrak{P}^p\{B_i\}$ with $B_i=B_0$ for all i. The space $L^p(B_0)$ is the space of all Bochner integrable functions[4] f on, say, $[0, 1]$ with values in B_0 and with $\|f\|=[\int_0^1\|f(x)\|^p dx]^{1/p}<\infty$. Boas' conjecture is verified by the next theorem.

THEOREM 2. *$l^p(B_0)$ and $L^p(B_0)$ are uniformly convex if (and, obviously, only if) B_0 is.*

Both of these results follow from the remaining theorem of this note. If all B_i, $i<\infty$, are uniformly convex there exists for each ϵ, $0<\epsilon\leqq 2$, a positive number $\delta_i(\epsilon)$ such that $\|b_i\|=\|b_i'\|=1$ and $\|b_i-b_i'\|>\epsilon$ implies $\|b_i+b_i'\|<2(1-\delta_i(\epsilon))$. The sequence $\{B_i, i<\infty\}$ will be said to have a common modulus of convexity if there is one function $\delta(\epsilon)>0$ which can be used here in place of all $\delta_i(\epsilon)$. It is clear that if we define $\delta_i(\epsilon)=\frac{1}{2}\inf\,[2-\|b_i+b_i'\|]$, where the infimum is taken over b_i, b_i' with $\|b_i\|=\|b_i'\|=1$ and $\|b_i-b_i'\|>\epsilon$, then such a $\delta(\epsilon)$ exists if and only if $\inf_i\delta_i(\epsilon)>0$ for every $\epsilon>0$. It is clear that $\delta(\epsilon)$ may be assumed to be a non-decreasing function of ϵ.

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[1] Presented to the Society, November 23, 1940.

[2] R. P. Boas, Jr., *Some uniformly convex spaces*, this Bulletin, vol. 46 (1940), pp. 304–311.

[3] M. M. Day, *Reflexive Banach spaces not isomorphic to uniformly convex spaces*, this Bulletin, vol. 47 (1941), pp. 313–317.

[4] S. Bochner, *Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind*, Fundamenta Mathematicae, vol. 20 (1933), pp. 262–276.