ON THE MAPPING OF QUADRATIC FORMS[1]

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The development of this paper was suggested by a theorem proposed by Bliss, proved by Albert,[2] by Reid,[3] and generalized by Hestenes and McShane.[4] That theorem had to do with two quadratic forms $P(z)$ and $Q(z)$ in real variables $z^1, z^2, \cdots, z^n$ with real coefficients, and may be stated as follows:

If $P(z)$ is positive at each point $z \neq (0)$ at which $Q(z)=0$, then there is a real number μ such that the quadratic form $P(z)+\mu Q(z)$ is positive definite.[5]

If one considers the set of points $\mathfrak{M}$ in the xy-plane into which the z-space is mapped by the transformation

$$x = P(z), \qquad y = Q(z), \tag{1}$$

he will note that the above theorem may be interpreted as asserting the existence of a *supporting line* of the map $\mathfrak{M}$ which has contact with $\mathfrak{M}$ only at $(x, y)=(0, 0)$. This suggests that the theorem is related to the theory of convex sets.

In the present paper it is proven (Theorem 1) that $\mathfrak{M}$ is a *convex* set. Furthermore it is proven (Theorem 2) that if $P(z)$ and $Q(z)$ have no common zero except $z=(0)$, then $\mathfrak{M}$ is *closed*, and is either the entire xy-plane or an angular sector of angle less than π. Immediate corollaries include not only the theorem quoted above, but also statements of criteria for the existence of (1) semi-definite, and (2) definite linear combinations $\lambda P(z)+\mu Q(z)$. The author hopes in a subsequent paper to obtain analogous results for the general case of m quadratic forms.

Throughout the paper it is to be understood without further statement that $P(z)$ and $Q(z)$ are quadratic forms in $z^1, z^2, \cdots, z^n$, with real coefficients, the variables z^i being restricted to real values.

1. **Convexity, and the condition for $\lambda P(z)+\mu Q(z) \geqq 0$.** We give first the following theorem.

[1] Presented to the Society, December 31, 1940.

[2] This Bulletin, vol. 44 (1938), p. 250.

[3] This Bulletin, vol. 44 (1938), p. 437.

[4] Transactions of this Society, vol. 47 (1940), p. 501.

[5] While the present paper was in press, Professor N. H. McCoy kindly called the author's attention to the fact that this theorem was proven first by Paul Finsler: *Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formen*, Commentarii Mathematici Helvetici, vol. 9 (1937), pp. 188–192. Apparently this work had been overlooked by the authors referred to above.