CONDITIONS FOR THE CONTINUITY OF ARC-PRESERVING TRANSFORMATIONS[1]

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1. **Introduction.** A single-valued transformation $T(A)=B$, where A and B are topological spaces, is said to be *arc-preserving*[2] provided that the image of every simple arc in A is either a simple arc or a a single point in B. Even when A is a simple arc, an arc-preserving transformation may fail to be continuous; for example: on the unit interval $(x_0=0\leqq x\leqq 1=x_1)$ let $x_n=1/n$ $(n=1, 2, 3, \cdots)$. Define $T(x_0)=x_0$ and for each interval A_n $(x_{n+1}\leqq x\leqq x_n)$ let $T(A_n)=A$ be a topological transformation such that $T(x_n)=x_0$ or x_1 according as n is even or odd. Then the transformation $T(A)=A$ is arc-preserving, but fails to be continuous at x_0.

The results of this paper concern conditions under which an arc-preserving transformation is continuous, and the conclusions lead to homeomorphisms. We consider only the case where A is a locally connected continuum. The transformation T may be made continuous by putting conditions on the space A or by putting added conditions on the transformation T itself. In this paper we take both points of view. We shall say that A is *strongly arcwise connected* provided every infinite subset of A intersects some arc of A in infinitely many points. Our principal theorem states that if A is cyclic and $T(A)=B$ is arc-preserving then T will be topological or B will be an arc provided either A is strongly arcwise connected or T is tree-preserving[3] (that is, the image of every tree in A is a tree or a single point in B). Moreover, we show that if B is not an arc then A must be strongly arcwise connected in order that a topological mapping be the only arc-preserving transformation of A onto B.

Throughout the paper A is a locally connected continuum and T is a single-valued transformation, but not necessarily continuous. It is understood that a single point is to be regarded as an arc.

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[2] See G. T. Whyburn, *Arc-preserving transformations*, American Journal of Mathematics, vol. 58 (1936), pp. 305–312. See also D. W. Hall and G. T. Whyburn, *Arc- and tree-preserving transformations*, Transactions of this Society, vol. 48 (1940), pp. 63–71.

[3] See R. G. Simond, Duke Mathematical Journal, vol. 4 (1938), pp. 575–589; also Hall and Whyburn, loc. cit.