ON SPHERICAL CYCLES[1]

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Given a metric separable space Y, we consider the homology group $B^n(Y)$ obtained using n-dimensional singular cycles in Y with integer coefficients. Every continuous mapping $f \in Y^{S^n}$ of the oriented n-dimensional sphere S^n into Y defines uniquely an element $h(f)$ of $B^n(Y)$. Clearly if $f_0, f_1 \in Y^{S^n}$ are two homotopic mappings, then $h(f_0) = h(f_1)$.

The homology classes $h(f)$ will be called *spherical homology classes*. A cycle will be called *spherical* if its homology class is spherical.[2]

THEOREM 1. *If Y is arcwise connected, the spherical homology classes form a subgroup of $B^n(Y)$.*

Let $p \in S^n$, $q \in Y$, and let $S^n = S^n_+ + S^n_-$ be a decomposition of S^n into two hemispheres such that $p \in S^n_+ \cdot S^n_-$. Consider $f_0, f_1 \in Y^{S^n}$. It is well known that, replacing if necessary f_0 and f_1 by homotopic mappings, we may assume that $f_0(S^n_+) = q$ and that $f_1(S^n_-) = q$. Defining $f = f_0$ on S^n_- and $f = f_1$ on S^n_+ we clearly have

$$f \in Y^{S^n}, \qquad h(f) = h(f_0) + h(f_1).$$

The homology class $h(f_0) + h(f_1)$ is therefore spherical.

Let M^r be an r-dimensional (finite or infinite) manifold[3] and P^{r-n-1} $(n > 0)$ an at most $(r-n-1)$-dimensional subpolyhedron of M^r.

THEOREM 2. *Every n-dimensional cycle γ^n in $M^r - P^{r-n-1}$ such that $\gamma^n \sim 0$ in M^r is spherical (with respect to $M^r - P^{r-n-1}$).*

Let a^{r-n-1} be an $(r-n-1)$-dimensional simplex of M^r and b^{n+1} the $(n+1)$-cell dual to it. The boundary ∂b^{n+1} is contained in $M^r - P^{r-n-1}$ and is a spherical cycle. Since $M^r - P^{r-n-1}$ is connected, the spherical homology classes of $B^n(M^r - P^{r-n-1})$ form a group. It follows that each cycle of the form

$$(*) \qquad \partial\left(\sum_i \alpha_i b_i^{n+1}\right)$$

is a spherical cycle with respect to $M^r - P^{r-n-1}$. The cycle γ^n is homologous in $M^r - P^{r-n-1}$ to a cycle of the form (*). Therefore γ^n is spherical.

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[1] Presented to the Society, April 13, 1940.

[2] Spherical cycles were considered by W. Hurewicz, Proceedings, Akademie van Wetenschappen te Amsterdam, vol. 38 (1935), pp. 521–528.

[3] See K. Reidemeister, *Topologie der Polyeder*, Leipzig, 1938, p. 151.