ON BIORTHOGONAL MATRICES[1]

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Introduction. Consider the basis consisting of a number system $\mathfrak{A}$ of type D, two general ranges $\mathfrak{P}^1$, $\mathfrak{P}^2$, and two positive hermitian matrices ϵ^1, ϵ^2. We introduce two binary relations for pairs of non-modular matrices. The matrices κ^{12}, ϕ^{21} are said to be contraceding as to $\epsilon^1\epsilon_\kappa^2$, ϵ^2 in case κ^{12}, ϕ^{21} are by columns of $\mathfrak{M}(\epsilon^1)$, $\mathfrak{M}(\epsilon^2)$ respectively and such that $J^{2\kappa}\kappa^{12}\mu^2 = J^2\phi^{*12}\mu^2$ for every μ^2 in the set $\mathfrak{M}(\epsilon_\kappa^2 \frown \epsilon^2)$. It is evident that when κ^{12} is of type $\mathfrak{M}(\epsilon^1)\overline{\mathfrak{M}}(\epsilon^2)$, then the contracedence property implies that $J^2\kappa^{12}\phi^{21} = \epsilon_\kappa^1$ but not conversely. The main results are stated in Theorems 2 and 3. We next consider ϵ_0^1, ϵ_1^1 both idempotent as to ϵ^1. Suppose that κ^{12} is by columns of $\mathfrak{M}(\epsilon_0^1)$ and ϕ^{21} is by rows-conjugate of $\mathfrak{M}(\epsilon_1^1)$. Take any pair of vectors μ^1, ν^1 modular as to ϵ_0^1, ϵ_1^1 respectively such that $J^1\kappa^{*21}\mu^1$, $J^1\phi^{21}\nu^1$ are in $\mathfrak{M}(\epsilon^2)$. If $J^1\bar{\mu}^1\nu^1$ is equal to the inner product $J^2(J^1\bar{\mu}^1\kappa^{12}, J^1\phi^{21}\nu^1)$, then κ^{12}, ϕ^{21} are said to be biorthogonal as to $\epsilon^1\epsilon_0^1\epsilon_1^1\epsilon^2$. When $\epsilon_1^1 = \epsilon_0^1$, then κ^{12}, ϕ^{21} are said to be biorthogonal as to $\epsilon^1\epsilon_0^1\epsilon^2$ in case they are biorthogonal as to $\epsilon^1\epsilon_0^1\epsilon_0^1\epsilon^2$. With proper restrictions imposed upon κ^{12}, ϕ^{21}, we obtain the contracedence property. In a later paper, we shall establish the relations of biorthogonality and a certain mode of interchange of integration processes.

1. **Preliminary results.** Consider the basis $\mathfrak{A}$, $\mathfrak{P}^1$, $\mathfrak{P}^2$, ϵ^1, and κ^{12} which is by columns of $\mathfrak{M}(\epsilon^1)$. E. H. Moore's generalized Fourier processes give $\epsilon_\kappa^2 \equiv J^1\kappa^{*21}\kappa^{12}$ and $\epsilon_\kappa^1 \equiv J^{2\kappa}\kappa^{12}\kappa^{*21}$. The spaces $\mathfrak{M}(\epsilon_\kappa^1)$ and $\mathfrak{M}(\epsilon_\kappa^2)$ are in one-to-one correspondence (denoted by $\leftrightarrow$) via the transformations $J^1\kappa^{*21}$ and $J^{2\kappa}\kappa^{12}$, and the correspondences are orthogonal in the sense that the moduli of the corresponding vectors are preserved.[2]

(A)[3] Suppose that $\mathfrak{M}_1(\epsilon_\kappa^1) \leftrightarrow \mathfrak{M}_1(\epsilon_\kappa^2)$ and $\mathfrak{M}_2(\epsilon_\kappa^1) \leftrightarrow \mathfrak{M}_2(\epsilon_\kappa^2)$ via the transformations $J^1\kappa^{*21}$, $J^{2\kappa}\kappa^{12}$. Then $\mathfrak{M}_1(\epsilon_\kappa^1)$ is a subset of $\mathfrak{M}_2(\epsilon_\kappa^1)$ if and only if $\mathfrak{M}_1(\epsilon_\kappa^2)$ is a subset of $\mathfrak{M}_2(\epsilon_\kappa^2)$; $\mathfrak{M}_1(\epsilon_\kappa^1)$ is linearly J^1-closed if and only if $\mathfrak{M}_1(\epsilon_\kappa^2)$ is linearly $J^{2\kappa}$-closed; and $\mathfrak{M}_1(\epsilon_\kappa^1)$ is everywhere dense in $\mathfrak{M}_2(\epsilon_\kappa^1)$ if and only if $\mathfrak{M}_1(\epsilon_\kappa^2)$ is everywhere dense in $\mathfrak{M}_2(\epsilon_\kappa^2)$.

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[2] For a concise outline of Moore's generalized Fourier theory and its related topics, *see* Moore, *General Analysis*, I, pp. 19–26. For an important classical instance, *see* E. Schmidt, *Über die Auflösung linearer Gleichungen mit unendlichvielen Unbekannten*, Rendiconti del Circolo Matematico di Palermo, vol. 25 (1908), pp. 56–77.

[3] For the demonstrations of the following results, *see* the author's forthcoming paper *On non-modular matrices*.