ELEMENTARY PROOF OF A THEOREM ON LORENTZ MATRICES

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Let x and y be real n- and m-vectors and x^2, y^2 the scalar squares of x, y. The corresponding Lorentz matrices are matrices of $(n+m)$-dimensional real linear transformations which leave the quadratic form x^2-y^2 invariant. Let the transformation be written in the form

$$(1) \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax+By \\ Cx+Dy \end{pmatrix}.$$

Then *the signs of the determinants* $|A|$ *and* $|D|$ *form two* 1-*dimensional representations of the Lorentz group*. Two algebraic proofs at present available for this fact[1] depend on a recursive factorization of the Lorentz matrix into simple factors or on deeper facts from the theory of representations. On the other hand, a simple topological proof may be given in quite an obvious manner. In this note the topological proof is briefly sketched and then a simple algebraic proof is given which does not depend on recursive factorization or representation theory and is valid in any real field.

The set defined by $x^2-y^2 \geqq 1$ in the real $(n+m)$-dimensional space possesses one basic $(n-1)$-dimensional (finite) cycle Γ which can most easily be represented by the $(n-1)$-dimensional basic cycle of the $(n-1)$-dimensional sphere $x^2=1$, $y=0$. Now Γ is transformed by (1) into a cycle homologous to $+\Gamma$ or to $-\Gamma$ according as $|A|$ is positive or negative. The formal proofs of these topological facts are obtained most easily from the remark that the whole space $x^2-y^2 \geqq 1$ can be retracted into its subset $x^2=1$, $y=0$ by a deformation which does not change the value of $x/(x^2)^{1/2}$ for any point. That *sign* $|A|$ is a one-dimensional representation of the Lorentz group is of course evident from the fact that Γ is transformed by (1) into a cycle homologous to *sign* $|A| \cdot \Gamma$. The statement concerning the signature of $|D|$ depends on a similar consideration for the set defined by $x^2-y^2 \leqq -1$.

Now let the elements of the matrix in (1) belong to any real field. Let the unit matrices of dimensions n and m be denoted by E_n and E_m. The fact that the matrix in (1) is a Lorentz matrix may be expressed by the relations:

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[1] Cf. W. Givens, *Factorization and signatures of Lorentz matrices*, this Bulletin, vol. 46 (1940), pp. 81–85, where other references are given. My thanks are due to Dr. Murnaghan who drew my attention to the above theorem.