and by the lemma and (18) for sufficiently large n

$$\Delta \leqq \frac{\epsilon}{2c_1} c_1 + \sum_{1 \leqq k \leqq n, |\theta - \theta_k^{(n)}| > \delta} \tfrac{1}{2} 2M \, | \, l_k[\theta - \pi/2n] + l_k[\theta + \pi/2n] \, |$$
$$< \epsilon/2 + MO(1/n) < \epsilon,$$

where $M = \max_{-1 \leqq x \leqq +1} |f(x)|$, and this proves our theorem.

BUDAPEST, HUNGARY

DISCONTINUOUS CONVEX SOLUTIONS OF DIFFERENCE EQUATIONS[1]

FRITZ JOHN

This paper contains some conditions for continuity of convex solutions of a difference equation.

A function $f(x)$ defined for $a \leqq x \leqq b$ is *convex*, if

$$\left(\frac{x+y}{2}\right) \leqq \frac{f(x) + f(y)}{2}. \tag{1}$$

If $f(x)$ is convex and bounded from above in $a \leqq x \leqq b$, then $f(x)$ is continuous (see Bernstein [1, p. 422]).[2] If $f(x)$ is convex in $a \leqq x \leqq b$ and y a fixed number with $a < y < b$, let the function $\phi_y(x)$ be defined by

$$\phi_y(x) = \lim_{\alpha \to x - y} f(y + \alpha),$$

where α assumes *rational* values only. Then $\phi_y(x)$ is uniquely defined, continuous, and convex for $a < x < b$ (F. Bernstein [1, p. 431, Theorem 7]); moreover $\phi_y(x) = f(x)$ for rational $y - x$.

THEOREM 1. *If there exists at most one continuous convex solution of the difference equation*

$$F(x, f(x), f(x+1), \cdots, f(x+n)) = g(x), \qquad x > 0, \tag{2}$$

where F and g are continuous functions of their arguments, then there exist no discontinuous convex solutions.

PROOF. If $f(x)$ is a convex solution, then, for $x - y$ rational,

$$F(x, \phi_y(x), \phi_y(x+1), \cdots, \phi_y(x+n)) = g(x);$$

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[1] Presented to the Society, September 12, 1940.

[2] The numbers in brackets refer to the bibliography.