ON A CONVERGENCE THEOREM FOR THE LAGRANGE INTERPOLATION POLYNOMIALS

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The unique polynomial of degree $(n-1)$ assuming the values $f(x_1), \cdots, f(x_n)$ at the abscissas $x_1, x_2, \cdots, x_n$, respectively, is given by the Lagrange interpolation formula

$$(1) \qquad L_n(f) = f(x_1)l_1(x) + f(x_2)l_2(x) + \cdots + f(x_n)l_n(x).$$

Here

$$(2) \qquad l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)}, \qquad k = 1, 2, \cdots, n$$

(fundamental polynomials of the Lagrange interpolation), and the polynomial $\omega(x)$ is defined by

$$(3) \qquad \omega(x) = c(x - x_1)(x - x_2) \cdots (x - x_n),$$

where c denotes an arbitrary constant not equal to zero. It is known and easy to verify that

$$(4) \qquad l_1(x) + l_2(x) + \cdots + l_n(x) \equiv 1.$$

In the Lagrange interpolation formula let

$$(5) \qquad x_k = x_k^{(n)} = \cos (2k - 1)\pi/2n = \cos \theta_k^{(n)}$$

which implies

$$(6) \qquad \omega(x) = T_n(x) = \cos (n \operatorname{arc} \cos x) = \cos n\theta, \qquad \cos \theta = x$$

(Tchebyscheff polynomial). In this case we have

$$(7) \qquad l_k(x) = l_k[\theta] = (-1)^{k+1} \frac{\cos n\theta \sin \theta_k^{(n)}}{n(\cos \theta - \cos \theta_k^{(n)})}, \qquad k = 1, 2, \cdots, n;\ x = \cos \theta;$$

and

$$(8) \qquad L_n(f) = L_n[f; \theta] = \sum_{k=1}^{n} f(\cos \theta_k^{(n)})(-1)^{k+1} \frac{\cos n\theta \sin \theta_k^{(n)}}{n(\cos \theta - \cos \theta_k^{(n)})}, \qquad x = \cos \theta.$$

Suppose $f(x)$ to be a continuous function; then it is known that